Lehmer-3 Mean Cordial Graphs

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Abstract :

Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A graph $G = (V, E)$ with p vertices and q edges is said to admit Lehmer-3 mean cordial labeling if the mapping h from $V(G)$ to $\{1, 2, 3\}$ induces the mapping h^* from $E(G)$ to $\{1,2,3\}$ as $h^*(xy) = \left\lceil \frac{h(x)^3+h(y)^3}{h(x)^2+h(y)^2} \right\rceil$ with the condition that the number of vertices labeled with i and the number of vertices labeled with j differ at most by 1, the number of edges labeled with i and the number of edges labeled with j differ at most by 1 where $i, j \in \{1,2,3\}$. A graph with a Lehmer-3 mean cordial labeling is called a Lehmer-3 mean cordial graph. In this paper, we prove that all trees, cycle graph $C_n (n \equiv 1,2 (mod\ 3))$, pinwheel graph PW_n, armed crown graph $AC_{m,n}$, the graphs $L_n \odot K_1$ and $CL_n \odot K_1$ are Lehmer-3 mean cordial graphs.

Keywords : *Labeling, Cordial Labeling, Mean Cordial Labeling*

1. Introduction

All graphs considered here are finite, simple, connected and undirected. Cordial labeling was introduced by Cahit [1] in 1987. The notion of mean cordial labeling was introduced by R. Ponraj, M. Sivakumar and M. Sundaram. Let h be a mapping from the vertex set of G to $\{0, 1, 2\}$. For each edge xy assign the label $\left\lceil \frac{h(x)+h(y)}{2} \right\rceil$. The mapping h is called a mean cordial labeling of G if the number of vertices labeled with i and the number of vertices labeled with j differ at most by 1, number of edges labeled with i and the number of edges labeled with j differ at most by 1 where $i, j \in \{1,2,3\}$. Motivated by this concept, we introduced a new labeling called Lehmer-3 mean cordial labeling. In this section we provide a summary of definitions required for our investigation.

Definition 1.1. The caterpillar graph CP_t is a tree with the property that a path P_t remains if all the pendant vertices are deleted.

Definition 1.2. The friendship graph F_n consists of n triangles with a common vertex. The pinwheel graph PW_n is obtained from the friendship graph F_n by identifying the outer edge of each triangle in F_n with an edge of a new triangle.

Definition 1.3. The ladder graph L_n is defined to be the cartesian product $P_n \times K_2$, where P_n is the path graph on n vertices and K_2 is the complete graph on two vertices.

Definition 1.4. The circular ladder graph CL_n is defined to be the cartesian product $C_n \times K_2$, where C_n is the cycle graph on n vertices and K_2 is the complete graph on two vertices.

Definition 1.5. An armed crown graph $AC_{m,n}$ is obtained by joining a vertex of degree one in the path P_n at each vertex of the cycle C_m by an edge.

Definition 1.6. The corona product $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Definition 1.7. Let p be a real number. Lehmer-p mean of positive real numbers $x_1, x_2, \dots, x_n$ is defined as $L(p) = \frac{\sum_{i=1}^{n} {x_i}^p}{\sum_{i=1}^{n} {x_i}^{p-1}}$.

Notation 1.8. $v_h(k)$ = the number of vertices labeled with k

$e_{h^*}(k)$ = the number of edges labeled with k.

Definition 1.9. Let G be a simple graph and let $h: V(G) \to \{1, 2, 3\}$. For each edge xy assign $h^*(xy) = \left\lceil \frac{h(x)^3 + h(y)^3}{h(x)^2 + h(y)^2} \right\rceil$. The mapping h is called a Lehmer-3 mean cordial labeling if $|v_h(i) - v_h(j)| \leq 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \leq 1$ for all $i, j \in \{1, 2, 3\}$. A graph with a Lehmer-3 mean cordial labeling is called a Lehmer-3 mean cordial graph.

Example 1.10. Figure 1 shows G_1 is a Lehmer-3 mean cordial graph, while G_2 is not a lehmer-3 mean cordial graph

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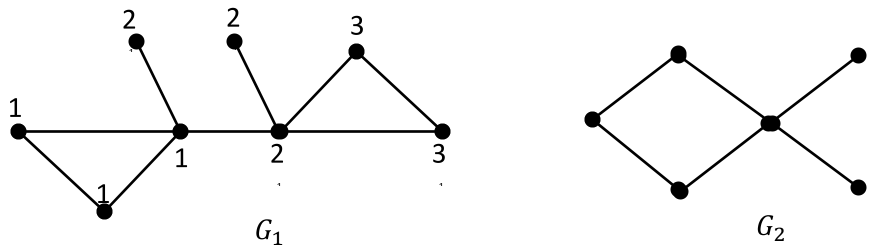

Figure 1

2. Main Results

Theorem 2.1. All trees are Lehmer-3 mean cordial graphs.

Proof. Let T be a tree with n vertices. Let $x_i (1 \leq i \leq n)$ be the vertices of T.

Suppose that T has a center say u. Take $u = x_1$. If r vertices are adjacent to u, denote these vertices by $x_2, x_3, \dots, x_{r+1}$. If $k_1, k_2, \dots, k_r$ vertices are adjacent to $x_2, x_3, \dots, x_{r+1}$ respectively and are at distance two from u, denote these vertices by $x_{r+2}, \dots, x_{r+k_1+1}, x_{r+k_1+2}, \dots, x_{r+k_1+k_2+1}, \dots, x_{r+k_1+\cdots+k_r+1}$. Continue the same process to denote the vertices of T.

Suppose that T has two centers say u and v. Take either $u = x_1$ or $v = x_1$ and follow he process discussed above to denote the vertices of T.

Define $h: V(G) \rightarrow \{1, 2, 3\}$ by $h(x_i) = \begin{cases} 1 & if \ 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil \\ 2 & if \ \left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq \left\lceil \frac{2n}{3} \right\rceil \\ 3 & if \ \left\lceil \frac{2n}{3} \right\rceil + 1 \leq i \leq n \end{cases}$

Then the number of vertices and edges labeled with $i \in \{1, 2, 3\}$ are as follows:

Case(i):$n \equiv 0(mod\ 3)$

$$v_h(i) = \frac{n}{3}, e_{h^*}(1) = \left\lfloor \frac{n-1}{3} \right\rfloor, e_{h^*}(2) = e_{h^*}(3) = \left\lceil \frac{n-1}{3} \right\rceil.$$

Case(ii):$n \equiv 1(mod\ 3)$

$$v_h(1) = \left\lceil \frac{n}{3} \right\rceil, v_h(2) = v_h(3) = \left\lfloor \frac{n}{3} \right\rfloor \text{ and } e_{h^*}(i) = \frac{n-1}{3}.$$

Case(iii):$n \equiv 2(mod\ 3)$

$$v_h(1) = v_h(2) = \left\lceil \frac{n}{3} \right\rceil, v_h(3) = \left\lfloor \frac{n}{3} \right\rfloor, e_{h^*}(1) = e_{h^*}(3) = \left\lfloor \frac{n-1}{3} \right\rfloor \text{ and } e_{h^*}(2) = \left\lceil \frac{n-1}{3} \right\rceil.$$

In each case $|v_h(i) - v_h(j)| \leq 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \leq 1$ for all $i, j \in \{1, 2, 3\}$. Hence T is a Lehmer-3 mean cordial graph.

Theorem 2.2. The cycle graph C_n is Lehmer-3 mean cordial graph iff $n \equiv 1\ or\ 2(mod\ 3)$.

Proof. Suppose that $n \equiv 1\ or\ 2(mod\ 3)$. Let $x_i (1 \leq i \leq n)$ be the vertices of C_n and let $E(G) = \{x_i x_{i+1}: 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. Label the vertices of C_n as in Theorem 2.1. Then the number of vertices labeled with $i \in \{1, 2, 3\}$ are same as in Theorem 2.1. Also the number of edges labeled with $i \in \{1, 2, 3\}$ are as follows:

Case(i): $n \equiv 1(mod\ 3)$

$e_{h^*}(1) = e_{h^*}(2) = \left\lfloor \frac{n}{3} \right\rfloor$ and $e_{h^*}(3) = \left\lceil \frac{n}{3} \right\rceil$.

Case(ii): $n \equiv 2(mod\ 3)$

$e_{h^*}(1) = \left\lfloor \frac{n}{3} \right\rfloor$ and $e_{h^*}(2) = e_{h^*}(3) = \left\lceil \frac{n}{3} \right\rceil$.

In both cases $|v_h(i) - v_h(j)| \leq 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \leq 1$ for all $i, j \in \{1, 2, 3\}$.

Hence $C_n (n \equiv 1,2(mod\ 3))$ is a Lehmer-3 mean cordial graph.

Let $n \equiv 0(mod\ 3)$. Suppose that C_n admits Lehmer-3 mean cordial labeling. Then we have $v_h(i) = \frac{n}{3}$ for all $i \in \{1, 2, 3\}$. To obtain the edge conditions we must label the vertices as in Theorem 2.1. Then we have $e_{h^*}(1) = \frac{n}{3} - 1$ and $e_{h^*}(3) = \frac{n}{3} + 1$. Thus $|e_{h^*}(1) - e_{h^*}(3)| > 1$. Hence C_n is not a Lehmer-3 mean cordial graph when $n \equiv 0(mod\ 3)$.

Theorem 2.3. The complete graph K_n is a Lehmer-3 mean cordial graph iff $n \leq 2$.

Proof. Let $n \leq 2$. Then the result follows from Theorem 2.1.

Suppose that $K_n (n \geq 3)$ admits Lehmer-3 mean cordial labeling. Then $|v_h(i) - v_h(j)| \leq 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \leq 1$ for all $i, j \in \{1, 2, 3\}$.

Case(i): $n \equiv 0(mod\ 3)$

Let $n = 3k$, where k is any positive integer. Then $v_h(i) = k$ for all $i \in \{1, 2, 3\}$, $e_{h^*}(1) = \frac{k(k-1)}{2}$, $e_{h^*}(2) = \frac{k(3k-1)}{2}$ and $e_{h^*}(3) = \frac{k(5k-1)}{2}$.

Here $|e_{h^*}(i) - e_{h^*}(j)| > 1$ for all $k \geq 2$ and $i \neq j$, which is a contradiction. If $k = 1$, $|e_{h^*}(1) - e_{h^*}(3)| = 2$, which is a contradiction.

Case(ii):$n \equiv 1(mod\ 3)$

Let $n = 3k + 1$, where k is any positive integer.

Subcase(i):$v_h(1) = k + 1, v_h(2) = v_h(3) = k$

Here $e_{h^*}(1) = \frac{k(k+1)}{2}$, $e_{h^*}(2) = \frac{k(3k+1)}{2}$ and $e_{h^*}(3) = \frac{k(5k+1)}{2}$.

Subcase(ii):$v_h(2) = k + 1, v_h(1) = v_h(3) = k$

Here $e_{h^*}(1) = \frac{k(k-1)}{2}$, $e_{h^*}(2) = \frac{3k(k+1)}{2}$ and $e_{h^*}(3) = \frac{k(5k+1)}{2}$.

Subcase(iii):$v_h(3) = k + 1, v_h(1) = v_h(2) = k$

Here $e_{h^*}(1) = \frac{k(k-1)}{2}$, $e_{h^*}(2) = \frac{k(3k-1)}{2}$ and $e_{h^*}(3) = \frac{5k(k+1)}{2}$.

In each subcase $|e_{h^*}(i) - e_{h^*}(j)| > 1$ for all $k \geq 2$ and $i \neq j$, which is a contradiction. If $k = 1$, $|e_{h^*}(1) - e_{h^*}(3)| > 1$, which is a contradiction.

Case(iii):$n \equiv 2(mod\ 3)$

Let $n = 3k + 2$, where k is any positive integer.

Subcase(i): $v_h(1) = v_h(2) = k + 1, v_h(3) = k$

Here $e_{h^*}(1) = \frac{k(k+1)}{2}$, $e_{h^*}(2) = \frac{(k+1)(3k+2)}{2}$ and $e_{h^*}(3) = \frac{k(5k+3)}{2}$.

Subcase(ii):$v_h(1) = v_h(3) = k + 1, v_h(2) = k$

Here $e_{h^*}(1) = \frac{k(k+1)}{2}$, $e_{h^*}(2) = \frac{k(3k+1)}{2}$ and $e_{h^*}(3) = \frac{(k+1)(5k+2)}{2}$.

Subcase(iii):$v_h(2) = v_h(3) = k + 1, v_h(1) = k$

Here $e_{h^*}(1) = \frac{k(k-1)}{2}$, $e_{h^*}(2) = \frac{3k(k\mp1)}{2}$ and $e_{h^*}(3) = \frac{(k+1)(5k+2)}{2}$.

In each subcase $|e_{h^*}(i) - e_{h^*}(j)| > 1$ for all $k \geq 3$ and $i \neq j$, which is a contradiction. If $k = 1\ or\ 2$, $|e_{h^*}(1) - e_{h^*}(3)| > 1$, which is a contradiction.

Hence K_n is not a Lehmer-3 mean cordial graph for all $n \geq 3$.

Theorem 2.4. The pinwheel graph PW_n is a Lehmer-3 mean cordial graph.

Proof. Let G be the pinwheel graph PW_n. Let x, x_i, y_i, z_i $(1 \leq i \leq n)$ be the vertices of G and let $xx_i, xy_i, x_iy_i, x_iz_i, y_iz_i$ $(1 \leq i \leq n)$ be the edges of G. Define $h: V(G) \to \{1, 2, 3\}$ as follows:

Case(i):$n \equiv 0\ (mod\ 3)$

Let $n = 3k$, where k is any positive integer. Define $h: V(G) \to \{1, 2, 3\}$ by $h(x) = 1$ and

$$h(x_i) = h(y_i) = h(z_i) = \begin{cases} 1 & if \ \ 1 \le i \le k \\ 2 & if \ \ k+1 \le i \le 2k \\ 3 & if \ \ 2k+1 \le i \le 3k \end{cases}$$

Here the number of vertices and edges labeled with $i \in \{1,2,3\}$ are $v_h(1) = \left\lceil \frac{3n+1}{3} \right\rceil$, $v_h(2) = v_h(3) = \left\lfloor \frac{3n+1}{3} \right\rfloor$ and $e_{h^*}(i) = \frac{5n}{3}$.

Case(ii): $n \equiv 1 \ (mod\ 3)$

Let $n = 3k + 1$. Label the vertices of G as in case(i) for $1 \le i \le 3k$. Also label the vertices x_n, y_n, z_n by 1, 2, 3 respectively. Then the number of vertices and edges labeled with $i \in \{1,2,3\}$ are $v_h(1) = \left\lceil \frac{3n+1}{3} \right\rceil$, $v_h(2) = v_h(3) = \left\lfloor \frac{3n+1}{3} \right\rfloor$, $e_{h^*}(1) = \left\lfloor \frac{5n}{3} \right\rfloor$ and $e_{h^*}(2) = e_{h^*}(3) = \left\lceil \frac{5n}{3} \right\rceil$.

Case(iii): $n \equiv 2 \ (mod\ 3)$

Let $n = 3k + 2$. Label the vertices of G as in case(i) for $1 \le i \le 3k$. Also label the vertices $x_{n-1}, y_{n-1}, z_{n-1}$ by $1, 1, 2$ respectively and x_n, y_n, z_n by 2, 3, 3 respectively.

Then the number of vertices and edges labeled with $i \in \{1,2,3\}$ are $v_h(1) = \left\lceil \frac{3n+1}{3} \right\rceil$, $v_h(2) = v_h(3) = \left\lfloor \frac{3n+1}{3} \right\rfloor$, $e_{h^*}(1) = e_{h^*}(2) = \left\lfloor \frac{5n}{3} \right\rfloor$ and $e_{h^*}(3) = \left\lceil \frac{5n}{3} \right\rceil$.

In each case $|v_h(i) - v_h(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \le 1$ for all $i, j \in \{1, 2, 3\}$. Hence G admits Lehmer-3 mean cordial labeling.

Theorem 2.5 . The graph $L_n \odot K_1$ is a Lehmer-3 mean cordial graph.

Proof. Le G be the graph $L_n \odot K_1$. Let $x_i, y_i (1 \le i \le n)$ be the vertices of L_n and $x_i{}', \ y_i{}'(1 \le i \le n)$ be the added vertices to form G. Define $h: V(G) \to \{1, 2, 3\}$ as follows:

Case(i): $n \equiv 0 (mod\ 3)$

$$h(x_i) = h(y_i) = \begin{cases} 1 & if \ \ 1 \le i \le \frac{n}{3} + 1 \\ 2 & if \ \ \frac{n}{3} + 2 \le i \le \frac{2n}{3} \\ 3 & if \ \ \frac{2n}{3} + 1 \le i \le n \end{cases}$$

$$h(x_i') = h(y_i') = \begin{cases} 1 & if\ 1 \le i \le \dfrac{n}{3} - 1 \\ 2 & if\ \dfrac{n}{3} \le i \le \dfrac{2n}{3} \\ 3 & if\ \dfrac{2n}{3} + 1 \le i \le n \end{cases}$$

Here the number of vertices and edges labeled with $i \in \{1,2,3\}$ are $v_h(i) = \frac{4n}{3}$, $e_{h^*}(1) = e_{h^*}(2) = \left\lfloor \frac{5n-2}{3} \right\rfloor$ and $e_{h^*}(3) = \left\lceil \frac{5n-2}{3} \right\rceil$.

Case(ii): $n \equiv 1 (mod\ 3)$

$$h(x_i) = h(y_i) = \begin{cases} 1 & if\ 1 \le i \le \left\lceil \dfrac{n}{3} \right\rceil \\ 2 & if\ \left\lceil \dfrac{n}{3} \right\rceil + 1 \le i \le \left\lceil \dfrac{2n}{3} \right\rceil \\ 3 & if\ \left\lceil \dfrac{2n}{3} \right\rceil + 1 \le i \le n \end{cases}$$

$$h(x_i') = \begin{cases} 1 & if\ 1 \le i \le \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le \left\lceil \frac{2n}{3} \right\rceil \\ 3 & if\ \left\lceil \frac{2n}{3} \right\rceil + 1 \le i \le n \end{cases} \quad \text{and} \quad h(y_i') = \begin{cases} 1 & if\ 1 \le i \le \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 3 & if\ \left\lceil \frac{2n}{3} \right\rceil \le i \le n \end{cases}$$

Here $v_h(1) = \left\lceil \frac{4n}{3} \right\rceil, v_h(2) = v_h(3) = \left\lfloor \frac{4n}{3} \right\rfloor$ and $e_{h^*}(i) = \frac{5n-2}{3}$ for all $i \in \{1, 2, 3\}$.

Case(iii): $n \equiv 2 (mod\ 3)$

Label the vertices $x_i, y_i (1 \le i \le n)$ as in case(ii). Also label the vertices $x_i', y_i' (1 \le i \le n)$ by

$$h(x_i') = \begin{cases} 1 & if\ 1 \le i \le \left\lceil \frac{n}{3} \right\rceil \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 3 & if\ \left\lceil \frac{2n}{3} \right\rceil \le i \le n \end{cases} \quad \text{and} \quad h(y_i') = \begin{cases} 1 & if\ 1 \le i \le \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le \left\lceil \frac{2n}{3} \right\rceil - 1 \\ 3 & if\ \frac{2n}{3} \le i \le n \end{cases}$$

Here $v_h(1) = v_h(3) = \left\lceil \frac{4n}{3} \right\rceil, v_h(2) = \left\lfloor \frac{4n}{3} \right\rfloor$, $e_{h^*}(1) = \left\lfloor \frac{5n-2}{3} \right\rfloor$ and $e_{h^*}(2) = e_{h^*}(3) = \left\lceil \frac{5n-2}{3} \right\rceil$.

In each case $|v_h(i) - v_h(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \le 1$ for all $i, j \in \{1, 2, 3\}$. Hence G admits Lehmer-3 mean cordial labeling.

Theorem 2.6. The graph $CL_n \odot K_1$ is a Lehmer-3 mean cordial graph iff $n \ne 3$

Proof. Let G be the graph $CL_n \odot K_1$. Let $x_i, y_i (1 \le i \le n)$ be the vertices of CL_n and

x_i', $y_i'(1 \leq i \leq n)$ be the added vertices to form G.

Let $n = 3$. Then $|V(G)| = 6$ and $|E(G)| = 9$. Take $v_h(i) = 2$ for all $i, j \in \{1, 2, 3\}$. Then $e_{h^*}(1)$ is either 0 or 1 and so $|e_{h^*}(1) - e_{h^*}(2)| > 1$ or $|e_{h^*}(1) - e_{h^*}(3)| > 1$ or $|e_{h^*}(2) - e_{h^*}(3)| > 1$. Thus G is not a Lehmer-3 mean cordial graph.

Suppose that $n \neq 3$. Define $h: V(G) \to \{1, 2, 3\}$ as follows:

Case(i):$n \equiv 0(mod\ 3)$ and $n \neq 3$.

Subcase(i): n is odd

$h(x_i) = 1$ for $1 \leq i \leq n-2$, $h(x_{n-1}) = h(x_n) = 2$,

$$h(y_i) = \begin{cases} 1 & if\ \ 1 \leq i \leq \frac{n}{3} + 2 \\ 2 & if\ \ \frac{n}{3} + 3 \leq i \leq \frac{2n}{3} + 1 \\ 3 & if\ \ \frac{2n}{3} + 2 \leq i \leq n \end{cases}$$

$$h(x_i') = h(y_i') = \begin{cases} 2 & if\ \ 1 \leq i \leq \frac{n}{3} + 1 \\ 3 & if\ \ \frac{n}{3} + 2 \leq i \leq n \end{cases}$$

Subcase(ii): n is even

Label the vertices $x_i, y_i(1 \leq i \leq n)$ and x_i', $y_i'(i \neq \frac{n}{3} + 1)$ as in subcase(i). If $i = \frac{n}{3} + 1$, label the vertex x_i' by 2 and y_i' by 3.

In both subcases, the number of vertices and edges labeled with $i \in \{1,2,3\}$ are $v_h(i) = \frac{4n}{3}$ and $e_{h^*}(i) = \frac{5n}{3}$.

Case(ii):$n \equiv 1(mod\ 3)$

Subcase(i): n is even

$h(x_i) = 1$ for $1 \leq i \leq n$

$$h(y_i) = \begin{cases} 1 & if\ \ 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil \\ 2 & if\ \ \left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq \left\lceil \frac{2n}{3} \right\rceil \\ 3 & if\ \ \left\lceil \frac{2n}{3} \right\rceil + 1 \leq i \leq n \end{cases}$$

$$h(x_i') = h(y_i') = \begin{cases} 2 & if \ \ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 3 & if \ \ \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases}$$

Subcase(ii): n is odd

Label the vertices $x_i, y_i (1 \le i \le n)$ and x_i', $y_i' (i \ne \left\lceil \frac{n}{3} \right\rceil)$ as in subcase(i). If $i = \left\lceil \frac{n}{2} \right\rceil$, label the vertex x_i' by 2 and y_i' by 3.

In both subcases, $v_h(1) = \left\lceil \frac{4n}{3} \right\rceil$, $v_h(2) = v_h(3) = \left\lfloor \frac{4n}{3} \right\rfloor$, $e_{h^*}(1) = e_{h^*}(2) = \left\lceil \frac{5n}{3} \right\rceil$ and $e_{h^*}(3) = \left\lfloor \frac{5n}{3} \right\rfloor$.

Case(iii):$n \equiv 2(mod\ 3)$

Subcase(i): n is odd

$h(x_i) = 1$ for $1 \le i \le n$

$$h(y_i) = \begin{cases} 1 & if \ \ 1 \le i \le \left\lceil \frac{n}{3} \right\rceil \\ 2 & if \ \ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 3 & if \ \ \left\lceil \frac{2n}{3} \right\rceil \le i \le n \end{cases}$$

$$h(x_i') = h(y_i') = \begin{cases} 2 & if \ \ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 3 & if \ \ \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases}$$

Subcase(ii): n is even

Label the vertices $x_i, y_i (1 \le i \le n)$ and x_i', $y_i' (i \ne \left\lceil \frac{n}{3} \right\rceil + 1)$ as in subcase(i). If $i = \left\lceil \frac{n}{2} \right\rceil + 1$, label the vertex x_i' by 2 and y_i' by 3.

In both subcases, $v_h(1) = v_h(2) = \left\lceil \frac{4n}{3} \right\rceil$, $v_h(3) = \left\lfloor \frac{4n}{3} \right\rfloor$ $e_{h^*}(1) = e_{h^*}(2) = \left\lfloor \frac{5n}{3} \right\rfloor$ and $e_{h^*}(3) = \left\lceil \frac{5n}{3} \right\rceil$.

In each case, $|v_h(i) - v_h(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \le 1$ for all $i, j \in \{1, 2, 3\}$. Hence $CL_n \odot K_1$ admits Lehmer-3 mean cordial labeling.

Theorem 2.7. An armed crown graph $AC_{m,n}$ is a Lehmer-3 mean cordial graph.

Proof. Let $G = AC_{m,n}$. Let $x_i (1 \le i \le m)$ be the vertices of the cycle C_m and let

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$x_{ij}(1 \leq i \leq m; 1 \leq j \leq n)$ be the vertices of the paths attached at each x_i. Then $E(G) = \{x_i x_{i+1} : 1 \leq i \leq m-1\} \cup \{x_m x_1\} \cup \{x_i x_{i1} : 1 \leq i \leq m\} \cup \{x_{ij} x_{i(j+1)} : 1 \leq i \leq m; 1 \leq j \leq n-1\}$. Define $h: V(G) \to \{1, 2, 3\}$ as follows:

Case(i):$m \equiv 0(mod\ 3)$

Let $m = 3k$. Then $h(x_i) = 1$ for $1 \leq i \leq m$,

$$h(x_{ij}) = \begin{cases} 2 & if\ 1 \leq i \leq k \\ 3 & if\ k+1 \leq i \leq 2k \end{cases} \text{ for } 1 \leq j \leq n$$

$h(x_{ij}) = 1$ for $2k+1 \leq i \leq m$ and $1 \leq j \leq n-2$

$h(x_{(m-1)(n-1)}) = h(x_{m(n-1)}) = 2$ and $h(x_{(m-1)n}) = h(x_{mn}) = 3$

Here $v_h(i) = e_{h^*}(i) = \frac{m(n+1)}{3}$ for all $i \in \{1,2,3\}$.

Case(ii):$m \equiv 1(mod\ 3)$

Let $m = 3k+1$. Label the vertices $x_i(1 \leq i \leq m)$, $x_{ij}(1 \leq i \leq 3k; 1 \leq j \leq n)$ as in case(i) and the remaining vertices as follows:

Subcase(i):$n \equiv 0(mod\ 3)$

$$h(x_{mj}) = \begin{cases} 1 & if\ 1 \leq j \leq \frac{n}{3} \\ 2 & if\ \frac{n}{3}+1 \leq j \leq \frac{2n}{3} \\ 3 & if\ \frac{2n}{3}+1 \leq j \leq n \end{cases}$$

Here $v_h(1) = e_{h^*}(1) = \left\lceil \frac{m(n+1)}{3} \right\rceil$, $v_h(2) = e_{h^*}(2) = v_h(3) = e_{h^*}(3) = \left\lfloor \frac{m(n+1)}{3} \right\rfloor$

Subcase(ii): $n \equiv 1(mod\ 3)$

$$h(x_{mj}) = \begin{cases} 1 & if\ 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \leq j \leq \left\lceil \frac{2n}{3} \right\rceil \\ 3 & if\ \left\lceil \frac{2n}{3} \right\rceil + 1 \leq j \leq n \end{cases}$$

Here $v_h(1) = e_{h^*}(1) = v_h(2) = e_{h^*}(2) = \left\lceil \frac{m(n+1)}{3} \right\rceil$, $v_h(3) = e_{h^*}(3) = \left\lfloor \frac{m(n+1)}{3} \right\rfloor$

Subcase(iii):$n \equiv 2(mod\ 3)$

$$h(x_{mj}) = \begin{cases} 1 & if\ 1 \le j \le \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \le j \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 3 & if\ \left\lceil \frac{2n}{3} \right\rceil \le j \le n \end{cases}$$

Here $v_h(i) = e_{h^*}(i) = \frac{m(n+1)}{3}$ for all $i \in \{1,2,3\}$.

Case(iii): $m \equiv 2(mod\ 3)$

Let $m = 3k + 2$. Label the vertices $x_i (1 \le i \le m)$, $x_{ij} (1 \le i \le 3k; 1 \le j \le n)$ as in case(i) and the remaining vertices as follows:

Subcase(i): $n \equiv 0(mod\ 3)$

$$h(x_{(m-1)j}) = \begin{cases} 1 & if\ 1 \le j \le \frac{n}{3} - 1 \\ 2 & if\ \frac{n}{3} \le j \le n \end{cases} \quad \text{and} \quad h(x_{mj}) = \begin{cases} 1 & if\ 1 \le j \le \frac{n}{3} \\ 3 & if\ \frac{n}{3} + 1 \le j \le n \end{cases}$$

Here $v_h(1) = e_{h^*}(1) = v_h(2) = e_{h^*}(2) = \left\lceil \frac{m(n+1)}{3} \right\rceil$, $v_h(3) = e_{h^*}(3) = \left\lfloor \frac{m(n+1)}{3} \right\rfloor$.

Subcase(ii): $n \equiv 1,2(mod\ 3)$

$$h(x_{(m-1)j}) = \begin{cases} 1 & if\ 1 \le j \le \left\lfloor \frac{n}{3} \right\rfloor \\ 2 & if\ \left\lceil \frac{n}{3} \right\rceil \le j \le n \end{cases} \quad \text{and} \quad h(x_{mj}) = \begin{cases} 1 & if\ 1 \le j \le \left\lfloor \frac{n}{3} \right\rfloor \\ 3 & if\ \left\lceil \frac{n}{3} \right\rceil \le j \le n \end{cases}$$

Here $v_h(1) = e_{h^*}(1) = \left\lceil \frac{m(n+1)}{3} \right\rceil$, $v_h(2) = e_{h^*}(2) = v_h(3) = e_{h^*}(3) = \left\lfloor \frac{m(n+1)}{3} \right\rfloor$.

In each case $|v_h(i) - v_h(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \le 1$ for all $i, j \in \{1, 2, 3\}$. Hence G is a Lehmer-3 mean cordial graph.

Theorem 2.8. The graph $G(p, q)$ obtained by identifying the end vertices of the path P_t on the caterpillar graph CP_t is a Lehmer-3 mean cordial graph iff any one of the following hold:

(i) $s + t \equiv 0\ or\ 2\ (mod\ 3)$

(ii) $s + t \equiv 1\ (mod\ 3)$ and $s \ge 2(t - 1)$,

where s is the number of pendant vertices of G.

Proof. Let x_1 be the identified vertex of G. Let s be the number of pendant vertices of G. Then $p = q = s + t - 1$. Also G contains a cycle of length $t - 1$.

Case(i): $s + t \equiv 0\ or\ 2\ (mod\ 3)$

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If l_1 pendant vertices are adjacent to x_1, denote these vertices by $x_2, x_3, \ldots., x_{l_1+1}$. Let x_{l_1+2} be the vertex which is adjacent to x_1 on the cycle. If l_2 pendant vertices are adjacent to x_{l_1+2}, denote these vertices by $x_{l_1+3}, \ldots., x_{l_1+l_2+2}$. Let $x_{l_1+l_2+3}$ be the vertex which is adjacent to x_{l_1+2} on the cycle and continue the same process to denote the vertices of G. Define

$$h: V(G) \to \{1,2,3\} \text{ by } h(x_i) = \begin{cases} 1 & if \ 1 \le i \le \left\lceil \frac{p}{3} \right\rceil \\ 2 & if \ \left\lceil \frac{p}{3} \right\rceil + 1 \le i \le \left\lceil \frac{2p}{3} \right\rceil \\ 3 & if \ \left\lceil \frac{2p}{3} \right\rceil + 1 \le i \le n \end{cases}$$

Then the number of vertices and edges labeled with $i \in \{1,2,3\}$ are as follows:

Subcase(i):$p \equiv 1 (mod\ 3)$

$$v_h(1) = \left\lceil \frac{p}{3} \right\rceil, v_h(2) = v_h(3) = \left\lfloor \frac{p}{3} \right\rfloor, e_{h^*}(1) = e_{h^*}(2) = \left\lfloor \frac{p}{3} \right\rfloor \text{ and } e_{h^*}(3) = \left\lceil \frac{p}{3} \right\rceil.$$

Subcase(ii):$p \equiv 2 (mod\ 3)$

$$v_h(1) = v_h(2) = \left\lceil \frac{p}{3} \right\rceil, v_h(3) = \left\lfloor \frac{p}{3} \right\rfloor, e_{h^*}(1) = \left\lfloor \frac{p}{3} \right\rfloor \text{ and } e_{h^*}(2) = e_{h^*}(3) = \left\lceil \frac{p}{3} \right\rceil.$$

In each subcase $|v_h(i) - v_h(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| \le 1$ for all $i, j \in \{1,2,3\}$. Hence G is a Lehmer-3 mean cordial graph.

Case(ii):$s + t \equiv 1\ (mod\ 3)$ and $s \ge 2(t-1)$.

Let $x_1, x_2, , \ldots., x_{t-1}$ be the path on the cycle. If $k_1, k_2, \ldots., k_{t-1}$ pendant vertices are adjacent to $x_1, x_2, , \ldots., x_{t-1}$ respectively, denote these vertices by $x_t, x_{t+1}, \ldots., x_{t+k_1-1}, x_{t+k_1}, \ldots., x_{t+k_1+k_2-1}, \ldots., x_n$. Then label the vertices of G as in case(i).

Here $v_h(i) = e_{h^*}(i) = \frac{p}{3}$ for all $i \in \{1,2,3\}$. Hence G is a Lehmer-3 mean cordial graph.

Case(iii):$s + t \equiv 1\ (mod\ 3)$ and $s < 2(t-1)$.

Suppose that G admits Lehmer-3 mean cordial labeling. Then $v_h(i) = \frac{s+t-1}{3}$ for all $i \in \{1,2,3\}$. To obtain the edge conditions we must label the vertices as in case(i) or case(ii). In both cases, $e_{h^*}(1) = v_h(1) - 1$. Then $|e_{h^*}(1) - e_{h^*}(2)| > 1$ or $|e_{h^*}(1) - e_{h^*}(3)| > 1$ or $|e_{h^*}(2) - e_{h^*}(3)| > 1$, which is a contradiction. Hence G is not a Lehmer-3 mean cordial graph.

Example 2.9. The Lehmer-3 mean cordial labeling of the graph G obtained by identifying the end vertices of the path P_8 on the caterpillar graph CP_7 is shown in figure 2.

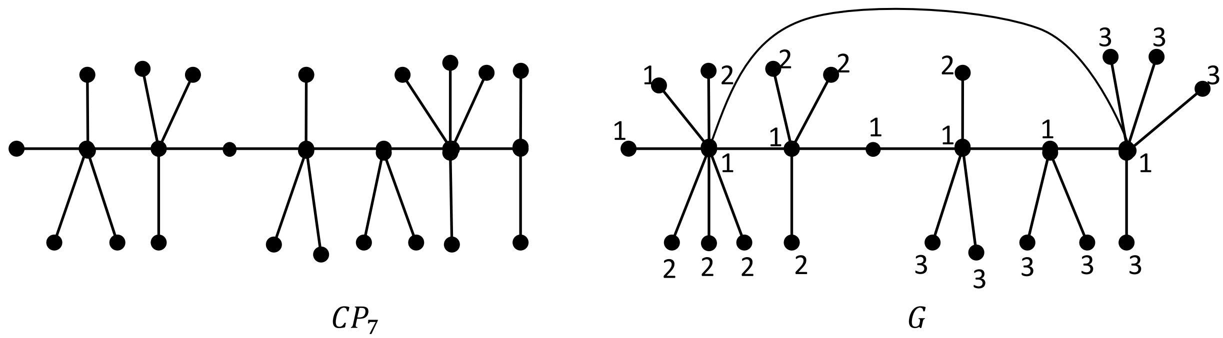

Figure 2

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