GENERALIZED CLOSED SETS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract

This paper is devoted to the study of intuitionistic fuzzy topological spaces. In this paper intuitionistic fuzzy generalized closed sets and intuitionistic fuzzy generalized open sets are introduced. We study some of their basic properties.

Key words: Intuitionistic fuzzy topology, intuitionistic fuzzy generalized closed sets, intuitionistic fuzzy generalized open sets, intuitionistic fuzzy $\mathcal{AT}_{1/2}$ space and intuitionistic fuzzy $\mathcal{AT}^{*}_{1/2}$ space.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [12] and later Atanassov [1] generalized this idea to intuitionistic fuzzy sets using the notion of fuzzy sets. On the other hand Coker [2] introduced intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. Krishnakumar and Senthilkumar [8] introduced generalized closed sets in topological spaces. In this paper we introduce intuitionistic fuzzy generalized closed sets and intuitionistic fuzzy generalized open sets and study some of their properties.

2. Preliminaries

Definition 2.1: [1] Let $X$ be a non empty fixed set. An intuitionistic fuzzy set (IFS in short) $A$ in $X$ is an object having the form

$$A = \{ (x, \mu_A (x), \nu_A (x)) / x \in X \}$$

where the functions $\mu_A (x): X \to [0,1]$ and $\nu_A (x): X \to [0,1]$ denote the degree of membership (namely $\mu_A (x)$) and the degree of non-membership (namely $\nu_A (x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A (x) + \nu_A (x) \leq 1$ for each $x \in X$. We denote the set of all intuitionistic fuzzy sets in $X$, by IFS ($X$).

Definition 2.2: [1] Let $A$ and $B$ be IFSs of the form

$$A = \{ (x, \mu_A (x), \nu_A (x)) / x \in X \}$$

and

$$B = \{ (x, \mu_B (x), \nu_B (x)) / x \in X \}.$$ 

Then

(a) $A \subseteq B$ if and only if $\mu_A (x) \leq \mu_B (x)$ and $\nu_A (x) \geq \nu_B (x)$ for all $x \in X$
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
(c) $A' = \{ (x, \nu_A (x), \mu_A (x)) / x \in X \}$
(d) $A \cap B = \{ (x, \max (\mu_A (x), \mu_B (x)), \min (\nu_A (x), \nu_B (x))) / x \in X \}$
(e) $A \cup B = \{ (x, \min (\mu_A (x), \mu_B (x)), \max (\nu_A (x), \nu_B (x))) / x \in X \}$

For the sake of simplicity, we shall use the notation $A = (x, \mu_A, \nu_A)$ instead of $A = \{ (x, \mu_A (x), \nu_A (x)) / x \in X \}$. Also for the sake of simplicity, we shall use the notation $A = \{ (x, \mu_A \mu_B, \nu_A \nu_B) / x \in X \}$ instead of $A = (x, (\mu_A \mu_B, (\nu_A, \nu_B))$. 

3857
The intuitionistic fuzzy sets $0_\sim = \{ \langle x, 0, 1 \rangle / x \in X \}$ and $1_\sim = \{ \langle x, 1, 0 \rangle / x \in X \}$ are respectively the empty set and the whole set of $X$.

**Definition 2.3:** [2] An *intuitionistic fuzzy topology* (IFT in short) on $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms.

1. $0_\sim, 1_\sim \in \tau$
2. $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$
3. $\bigcup G_i \in \tau$ for any family $\{ G_i / i \in J \} \subseteq \tau$.

In this case the pair $(X, \tau)$ is called an *intuitionistic fuzzy topological space* (IFTS in short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS in short) in $X$.

The complement $A^c$ of an IFOS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS in short) in $X$.

**Definition 2.4:** [2] Let $(X, \tau)$ be an IFTS and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in $X$. Then the intuitionistic fuzzy interior and an intuitionistic fuzzy closure are defined by

$$\text{int}(A) = \bigcup \{ G / G \text{ is an IFOS in } X \text{ and } G \subseteq A \},$$

$$\text{cl}(A) = \bigcap \{ K / K \text{ is an IFCS in } X \text{ and } A \subseteq K \}.$$ 

Note that for any IFS $A$ in $(X, \tau)$, we have $\text{cl}(A^c) = (\text{int}(A))^c$ and $\text{int}(A^c) = (\text{cl}(A))^c$.

**Definition 2.5:** [9] An IFS $A$ of an IFTS $(X, \tau)$ is an *intuitionistic fuzzy regular openset* (IFROS in short) if $A = \text{int}(\text{cl}(A))$, an *intuitionistic fuzzy regular closedset* (IFRCS in short) if $A = \text{cl}(\text{int}(A))$.

**Theorem 2.7:** [4] An IFS $A$ of an IFTS $(X, \tau)$ is an *intuitionistic fuzzy semi pre closedset* (IFSPCS in short) if there exists an IFPOS $B$ such that $B \subseteq A \subseteq \text{cl}(B)$.

The family of all IFSPCSs (respectively IFPOSs) of an IFTS $(X, \tau)$ is denoted by $\text{IFSPCS}(X)$ (respectively $\text{IFPOS}(X)$).

**Definition 2.8:** [9] An IFS $A$ of an IFTS $(X, \tau)$ is an *intuitionistic fuzzy generalized closedset* (IFGCS in short) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IFOS in $X$.

**Definition 2.9:** [7] An IFS $A$ of an IFTS $(X, \tau)$ is an *intuitionistic fuzzy generalized semi closedset* (IFGSCS in short) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IFOS in $X$.

**Definition 2.10:** [6] Let an IFS $A$ of an IFTS $(X, \tau)$. Then the semi closure of $A$ ($\text{scl}(A)$ in short) is defined as

$$\text{scl}(A) = \bigcap \{ K / K \text{ is an IFSCS in } X \text{ and } A \subseteq K \}.$$ 

**Definition 2.11:** [6] Let $A$ be an IFS of an IFTS $(X, \tau)$. Then the semi interior of $A$ ($\text{sint}(A)$ in short) is defined as

$$\text{sint}(A) = \bigcup \{ K / K \text{ is an IFOS in } X \text{ and } K \subseteq A \}.$$ 

**Definition 2.12:** [7] An IFS $A$ of an IFTS $(X, \tau)$ is an *intuitionistic fuzzy generalized semi closedset* (IFGSCS in short) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IFOS in $X$.

**Definition 2.13:** [7] Every IFPOS is IFOS in $(X, \tau)$.

**Definition 2.14:** [4] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS $(X, \tau)$ is said to be a

1. intuitionistic fuzzy semi closed set (IFCS in short) if $\text{int}(\text{cl}(A)) \subseteq A$,
2. intuitionistic fuzzy pre-closed set (IFPCS for short) if $\text{cl}(\text{int}(A)) \subseteq A$,
3. intuitionistic fuzzy $\alpha$-closed set (IF$\alpha$CS for short) if $\text{cl}(\text{cl}(A)) \subseteq A$. 

3858
(iv) intuitionistic fuzzy $\beta$-closed set (IF$\beta$CS for short) if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.

The respective complements of the above IFCSs are called their respective IFOSs.

The family of all IFSCs, IFPCSs, IF$\alpha$CSs and IF$\beta$CSs (respectively IF$\alpha$OSs, IFPOSs, IF$\alpha$OSs and IF$\beta$OSs) of an IFTS $(X,\tau)$ are respectively denoted by IFSC$(X)$, IFPC$(X)$, IF$\alpha$C$(X)$ and IF$\beta$C$(X)$ (respectively IFSO$(X)$, IFPO$(X)$, IF$\alpha$O$(X)$ and IF$\beta$O$(X)$).

**Definition 2.15:** [4] Two IFSs are said to be q-coincident (A \(\sim\) B in short) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

**Definition 2.16:** [4] Two IFSs are said to be not q-coincident (A \(\napprox\) B in short) if and only if $A \subseteq B^c$.

**Theorem 2.17:** [11] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS $(X, \tau)$ is said to be an

(i) intuitionistic fuzzy semi-pre closed set (IFSPCS for short) if there exists an IFPCS $B$ such that $\text{int}(B) \subseteq A \subseteq B$.

(ii) intuitionistic fuzzy semi-pre open set (IFSPOS for short) if there exists an intuitionistic fuzzy pre open set (IFPOS for short) $B$ such that $B \subseteq A \subseteq \text{cl}(B)$.

The family of all IFSPCSs (respectively IFSPOSs) of an IFTS $(X, \tau)$ is denoted by IFSPC$(X)$ (respectively IFSPO$(X)$).

Every IFSC (respectively IFOS) and every IFPCS (respectively IFPOS) is an IFSPCS (respectively IFSPOS). But the separate converses need not be true in general.

### 3. \(\tilde{A}\)-generalized closed sets in Intuitionistic fuzzy topological spaces

In this section we have introduced the notion of intuitionistic fuzzy \(\tilde{A}\)-generalized closed sets and studied some of their basic properties. Also we have provided the relationship between intuitionistic fuzzy \(\tilde{A}\)-generalized semi-pre closed sets with other intuitionistic fuzzy sets.

**Definition 3.1:** An IFS $A$ in an IFTS $(X, \tau)$ is said to be an intuitionistic fuzzy \(\tilde{A}\)-generalized semi-pre closed set (IF\(\tilde{A}\)GCS for short) if $\text{int}(\text{cl}(\text{int}(A))) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IFOS in $(X, \tau)$. The family of all IF\(\tilde{A}\)GCSs of an IFTS $(X, \tau)$ is denoted by IF\(\tilde{A}\)GC$(X)$.

**Example 3.2:** Let $X = \{ a, b \}$ and $G = \langle x, (0.4, 0.3), (0.4, 0.5) \rangle$. Then $\tau = \{ 0., G, 1. \}$ is an IFT on $X$ and the IFS $A = \langle x, (0.3, 0.1), (0.5, 0.6) \rangle$ is an IF\(\tilde{A}\)GCS in $(X, \tau)$.

**Theorem 3.3:** Every IFCS is an IF\(\tilde{A}\)GCS but not conversely.

**Proof:** Let $A$ be an IFCS in $X$ and let $A \subseteq U$ and $U$ is an IFOS in $(X, \tau)$. Since $\text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(A)$ and $A$ is an IFCS in $X$, int(cl(int(A))) \subseteq cl(A) = A \subseteq U$. Therefore $A$ is an IF\(\tilde{A}\)GCS in $X$.

**Example 3.4:** Let $X = \{ a, b \}$ and let $\tau = \{ 0., G, 1. \}$ is an IFT on $X$, where $G = \langle x, (0.2, 0.2), (0.7, 0.5) \rangle$. Then the IFS $A = G$ is an IF\(\tilde{A}\)GCS but not an IFCS in $X$.

**Theorem 3.5:** Every IFPCS is an IF\(\tilde{A}\)GCS but not conversely.

**Proof:** Let $A$ be an IFPCS in $X$ and let $A \subseteq U$ and $U$ is an IFOS in $(X, \tau)$. Since $\text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(A) \subseteq U$ by hypothesis, $A$ is an IF\(\tilde{A}\)GCS in $X$.

**Example 3.6:** Let $X = \{ a, b \}$ and let $\tau = \{ 0., G, 1. \}$ be an IFT on $X$, where $G = \langle x, (0.4, 0.3), (0.5, 0.6) \rangle$. Then the IFS $A = G$ is an IF\(\tilde{A}\)GCS but not an IFGCS in $X$ since $A \subseteq T$ but $\text{cl}(A) = \langle x, (0.5, 0.6), (0.4, 0.3) \rangle \supseteq G$.

**Theorem 3.7:** Every IFSPCS is an IF\(\tilde{A}\)GCS but not conversely.
**Proof:** Let A be an IFSPCS and A \( \subseteq U \) and U is an IF\(\tau\)OS in \((X, \tau)\). Since \(\text{int} (\text{cl} (\text{int}(A))) = A\) and \(A \subseteq U\), we have \(\text{int} (\text{cl} (\text{int}(A))) \subseteq U\). Therefore, A is an IF\(\alpha\)GCS.

**Example 3.8:** Let \(X = \{ a, b \}\) and let \(\tau = \{ 0, G, 1 \}\) where \(G = (x, (0.6, 0.7), (0.3, 0.2))\). Then the IFS \(A = (x, (0.7, 0.8), (0.1, 0.2))\) is an IF\(\alpha\)GCS in \((X, \tau)\) but not an IFSPCS in \((X, \tau)\).

**Theorem 3.9:** Every IF\(\beta\)CS is an IF\(\alpha\)GCS but not conversely.

**Proof:** Let A be an IF\(\beta\)CS and A \(\subseteq U\), U is an IFOS in \((X, \tau)\). Since \(\beta\text{cl}(A) = A\) and \(A \subseteq U\), we have \(\beta\text{cl}(A) \subseteq U\). Hence A is an IF\(\alpha\)GCS.

In Example 3.8, the IFS \(A = (x, (0.7, 0.8), (0.1, 0.2))\) is an IF\(\alpha\)GCS but not an IF\(\beta\)CS in \((X, \tau)\).

**Proposition 3.10:** Every IFSCS is an IF\(\alpha\)GCS but not conversely.

**Proof:** Let A be an IFSCS in X. Since every IFSCS is an IFSPCS, by Theorem 3.7, A is an IF\(\alpha\)GCS in X.

**Example 3.11:** Let \(X = \{ a, b \}\) and let \(\tau = \{ 0, G, 1 \}\), where \(G = (x, (0.2, 0.3), (0.5, 0.7))\). Then the IFS \(A = (x, (0.1, 0.2), (0.6, 0.8))\) is an IF\(\alpha\)GCS but not an IFSCS in \((X, \tau)\).

**Proposition 3.12:** Every IFPCS is an IF\(\alpha\)GCS but not conversely.

**Proof:** Since every IFPCS is an IFSPCS, the proof is obvious from Theorem 3.7.

**Example 3.13:** Let \(X = \{ a, b \}\) and let \(\tau = \{ 0, G_1, G_2, 1 \}\) where \(G_1 = (x, (0.4, 0.5), (0.4, 0.3))\) and \(G_2 = (x, (0.3, 0.1), (0.5, 0.6))\), then the IFS \(A = (x, (0.3, 0.1), (0.5, 0.6))\) is an IF\(\alpha\)GCS in \((X, \tau)\) but not an IFPCS in \((X, \tau)\).

**Proposition 3.14:** Every IF\(\alpha\)CS is an IF\(\alpha\)GCS but not conversely.

**Proof:** Let A be an IF\(\alpha\)CS. Since every IF\(\alpha\)CS is an IFSPCS, by Theorem 3.7, A is an IF\(\alpha\)GCS.

**Example 3.15:** Let \(X = \{ a, b \}\) and let \(\tau = \{ 0, G, 1 \}\) be an IFT on X, where \(G = (x, (0.7, 0.8), (0.3, 0.1))\). Then the IFS \(A = (x, (0.5, 0.4), (0.2, 0.3))\) is an IF\(\alpha\)GCS but not an IF\(\alpha\)CS in X since \(A \subseteq G\) but \(\text{cl} (\text{int} (\text{cl}(A))) = 1.\not\subseteq G\).

**Proposition 3.16:** Every IFRCS is an IF\(\alpha\)GCS but not conversely.

**Proof:** Let A be an IFRCS in X. By definition \(A = \text{cl} (\text{int}(A))\). This implies \(\text{cl}(A) = \text{cl}(\text{int}(A))\). Therefore, \(\text{cl}(A) = A\). Hence, A is an IFCS in X. By Theorem 3.3, A is an IF\(\alpha\)GCS in X.

**Example 3.17:** Let \(X = \{ a, b \}\) and let \(\tau = \{ 0, G, 1 \}\) be an IFT on X, where \(G = (x, (0.3, 0.4), (0.5, 0.6))\). The IFS \(A = T\) is an IF\(\alpha\)GCS but not an IFRCS in X since \(\text{cl} (\text{int}(A)) = (x, (0.3, 0.4), (0.5, 0.6)) \neq A\).

In the following diagram, we have provided relations between various types of intuitionistic fuzzy closedness.

![Diagram of intuitionistic fuzzy closedness relations]
The reverse implications are not true in general in the above diagram.

**Remark 3.18:** The intersection of two IFS\(\alpha\)GCSs is not an IFS\(\alpha\)GCS in general as seen in the following example.

**Example 3.19:** Let \(X = \{a, b\}\) and let \(\tau = \{0.1, G_1, G_2, G_3, G_4, 1.\}\) where \(G_1 = (x, (0.7, 0.8), (0.3, 0.2))\), \(G_2 = (x, (0.2, 0.1), (0.8, 0.9))\), \(G_3 = (x, (0.5, 0.6), (0.5, 0.4))\) and \(G_4 = (x, (0.6, 0.7), (0.4, 0.3))\), then the IFSs \(A = (x, (0.6, 0.7), (0.4, 0.2))\) and \(B = (x, (0.7, 0.6), (0.3, 0.4))\) are IFS\(\alpha\)GCSs in \((X, \tau)\) but \(A \cap B\) is not an IFS\(\alpha\)GCS in \((X, \tau)\). Since \(A \cap B = (x, (0.6, 0.6), (0.4, 0.4)) \subseteq G_4\) but \(\text{int}(\text{cl}(A \cap B)) = G_1 \subseteq G_4\).

**Theorem 3.20:** Let \((X, \tau)\) be an IFTS. Then for every \(A \in \text{IFS}(X)\) and for every \(B \in \text{IFS}(X)\), \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\) ⇒ \(B \in \text{IFS}(X)\).

**Proof:** Let \(B \subseteq U \subseteq \text{IFS}(X)\) be an IFS. Then since \(A \subseteq B \subseteq U\), by hypothesis, \(B \subseteq \text{int}(\text{cl}(\text{int}(A)))\). Therefore \(\text{int}(\text{cl}(\text{int}(B))) \subseteq \text{int}(\text{cl}(\text{int}(\text{int}(A)))) = \text{int}(\text{cl}(\text{int}(A))) \subseteq U\), since \(A \in \text{IFS}(X)\). Hence \(B \in \text{IFS}(X)\).

**Theorem 3.21:** An IFS \(A\) of an IFTS \((X, \tau)\) is an IFS\(\alpha\)GCS if and only if \(A \subseteq \text{int}(\text{cl}(\text{int}(A))) \Rightarrow A \in \text{IFS}(X)\).

**Proof:** Suppose that every IFS in \((X, \tau)\) is an IFS\(\alpha\)GCS. Let \(U \in \text{IFS}(X)\), then \(U \subseteq \text{int}(\text{cl}(\text{int}(U)))\). This implies \(\text{int}(\text{cl}(\text{int}(U))) = U\). Therefore \(U \in \text{IFS}(X)\). Hence \(\text{IFS}(X) \subseteq \text{IFS}(X)\). Let \(A \in \text{IFS}(X)\), then \(A \subseteq \text{IFS}(X) \subseteq \text{IFS}(X)\). Therefore \(A \in \text{IFS}(X)\). Hence \(\text{IFS}(X) \subseteq \text{IFS}(X)\). Thus \(\text{IFS}(X) = \text{IFS}(X)\).

**Sufficiency:** Suppose that \(\text{IFS}(X) = \text{IFS}(X)\). Let \(A \subseteq U \subseteq \text{IFS}(X)\). Then \(U \subseteq \text{IFS}(X)\) and \(\text{int}(\text{cl}(\text{int}(A))) \subseteq \text{int}(\text{cl}(\text{int}(U))) = U\), since \(U \in \text{IFS}(X)\), by hypothesis. Therefore \(A \in \text{IFS}(X)\) in \(X\).

**Theorem 3.22:** If \(A\) is an IFS\(\alpha\)GCS and \(A \subseteq \text{IFS}(X)\), then \(A \in \text{IFS}(X)\).

**Proof:** Since \(A \subseteq \text{IFS}(X)\) and \(A \in \text{IFS}(X)\), by hypothesis, \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\). But \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\). Therefore \(\text{int}(\text{cl}(\text{int}(A))) = A\). Hence \(A \in \text{IFS}(X)\).

**Theorem 3.24:** Let \(A\) be an IFS\(\alpha\)GCS in \((X, \tau)\) and \(c(\alpha, \beta)\) be an IFP in \(X\) such that \(c(\alpha, \beta) \subseteq \text{int}(\text{cl}(\text{int}(A)))\), then \(c(\alpha, \beta) \subseteq A\).

**Proof:** Let \(A\) be an IFS\(\alpha\)GCS and let \(c(\alpha, \beta) \subseteq \text{spcl}(A)\). If \(c(\alpha, \beta) \subseteq \text{cl}(\text{cl}(\alpha, \beta))\), then by Definition 2.16, \(A \subseteq \text{cl}(\text{cl}(\alpha, \beta))\). Since \(\text{cl}(\text{cl}(\alpha, \beta))\) is an IFS, then \(\text{cl}(\text{cl}(\alpha, \beta))\) is an IFS\(\alpha\)GCS. Then by hypothesis, \(\text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(\text{cl}(\alpha, \beta))\). Therefore by Definition 2.16, \(c(\alpha, \beta) \subseteq \text{spcl}(A)\), which is a contradiction to the hypothesis. Hence \(c(\alpha, \beta) \subseteq A\).

**Theorem 3.25:** For an IFS \(A\), the following conditions are equivalent.

\[(i)\] \(A\) is an IFS\(\alpha\)GCS and \(\text{IFS}(A)\)

\[(ii)\] \(A\) is an IFS\(\alpha\)ROS

**Proof:** \((i) \Rightarrow (ii)\) Let \(A\) be an IFS\(\alpha\)GCS and \(\text{IFS}(A)\). This implies that \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\). Since \(A\) is an IFS, \(\text{int}(A) = A\). Therefore \(\text{int}(\text{cl}(A)) \subseteq A\). Since \(A\) is an
IFOS, it is an IFPOS. Hence $A \subseteq \text{int} \left( \text{cl}(A) \right)$. Therefore $A = \text{int} \left( \text{cl}(A) \right)$. Hence $A$ is an IFROS. 

(ii) $\Rightarrow$ (i) Let $A$ be an IFROS. Therefore $A = \text{int} \left( \text{cl}(A) \right)$. Since every IFROS in an IFOS, $A$ is an IFOS and $A \subseteq A$. This implies $\text{int} \left( \text{cl}(A) \right) \subseteq A$. That is $\text{int} \left( \text{int}(A) \right) \subseteq A$. Therefore $A$ is an IFβCS. Hence by Theorem 3.9, $A$ is an IFβGCS.

**Theorem 3.26:** Let $F \subseteq A \subseteq X$ where $A$ is an IFOS and an IFβGCS in $X$. Then $F$ is an IFβGCS in $A$ if and only if $F$ is an IFβGCS in $X$.

**Proof:** Necessity: Let $U$ be an IFOS in $X$ and $F \subseteq U$. Also let $F$ be an IFβGCS in $A$. Then clearly $F \subseteq A \cap U$ and $A \cap U$ is an IFOS in $A$. Hence the semi-pre closure of $F$ in $A$, $\text{spcl}_A(F) \subseteq A \cap U$. By Theorem 3.23, $A$ is an IFSPCS. Therefore $\text{spcl}(F) = A$ and the semi-pre closure of $F$ in $X$, $\text{spcl}(F) \subseteq \text{spcl}(F) \cap A = \text{spcl}_A(F) \subseteq A \cap U \subseteq U$. That is $\text{spcl}(F) \subseteq U$ whenever $F \subseteq U$. Hence $F$ is an IFβGCS in $X$.

**Sufficiency:** Let $V$ be an IFOS in $A$ such that $F \subseteq V$. Since $A$ is an IFOS in $X$, $V$ is an IFOS in $X$. Therefore $\text{spcl}(F) \subseteq V$, since $F$ is an IFβGCS in $X$. Thus $\text{spcl}_A(F) = \text{spcl}(F) \cap A \subseteq V \cap A \subseteq V$. Hence $F$ is an IFβGCS in $A$.

**Theorem 3.27:** Let $(X, \tau)$ be an IFTS. Then for every $A \in \text{IFSPC}(X)$ and for every IFS $B$ in $X$, $\text{int}(A) \subseteq B \subseteq A$ if and only if $B \in \text{IFG}(X)$.

**Proof:** Let $A$ be an IFSPCS in $X$. Then by Definition 2.17, there exists an IFPCS, say $C$ such that $\text{int}(C) \subseteq A \subseteq C$. By hypothesis, $B \subseteq A$. Therefore $B \subseteq C$. Since $\text{int}(C) \subseteq A$, $\text{int} \left( \text{int}(A) \right)$ and $\text{int}(C) \subseteq B$. Thus $\text{int}(C) \subseteq B \subseteq C$ and by Definition 2.17, $B \in \text{IFSPC}(X)$. Hence by Theorem 3.7, $B \in \text{IFG}(X)$.

**References**


