ISSN- 2394-5125 VOL 7, ISSUE 11, 2020

SURVEY RESULTS ON DIFFERENCE EQUATIONS

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ABSTRACT: Brief descriptions of survey results on the field of difference equations are given. The articles on recent trends are analysed.

INTRODUCTION

The survey resultsarea special issue that contains 60 selected research articles. Theyfocus on the related papers to pave way for the future advancements. The subject discussed here brings out pertinent information regarding developments that are made in recent times. The paper presents a view on literature with advances ranging from 1960s. The concepts are observed from research papers and conference talks through online references. The equation which expresses a value of a sequence as the function of other term in the sequence is called a difference equation.An equation given by the form

$$f(n, y_n, \Delta y_n, \Delta^2 y_n, \dots,) = 0 \qquad (1)$$

where Δ is the forward difference operator defined by

$$\Delta y_n = y_{n+1} - y_n \qquad (2)$$

A real sequence$\{y_n\}$, defined for $n \in N_0 = \{0,1,2,3,\dots\}$ is the solution of (1).Mathematical computations are based on equations obtained from a given set of values that are discrete. The survey result combines theoretical and practical applications from divisions of oscillation, neutral, delay, fractional, stability and much more of the difference equations. Difference equations are applied in statistics, biology, dynamical system, economics and other fields.

1. **MAIN RESULT**

Three categories of difference equations are discussed in this paper:

1. The Difference Equations
2. Oscillation of difference equations
3. The Stability and Asymptotic behaviour of difference equations.

2.1 Survey Results on Difference Equations

Thedifference equation given as

$$y(n+1) = a(n)y(n), y(n_0) = y_0, \qquad n \geq n_0 \geq 0 \qquad (3)$$

$$y(n+1) = a(n)y(n) + g(n), y(n_0) = y_0, \; n \geq n_0 \geq 0 \qquad (4)$$

iscalled linear difference equation where $a(n)$ and $g(n)$ are real valued functions. The equation (3) is linear homogeneous first-order equation and (4) is non-homogeneous first-order equation.

In 1965 the Journal of Mathematical Analysis and Applications shows the work of SherwoodC. Chu and J. B. Diaz. The authors considered the solution $b(x)$for the equation,

$$b(x+1) - b(x) = g(x) \qquad (5)$$

The Euler's solution is applied to solve

$$G\big(b(x), b(x+1), \dots, b(x+n)\big) = g(x) \qquad (6)$$

where*G* is homogeneous of degree 1 and partial difference equation.

In the Journal of Economics Letters (1978), Lawrence J. LAU discussed thecompatibility of the followingequation,

ISSN- 2394-5125 VOL 7, ISSUE 11, 2020

$Y_t = RY_{t-1} + r_0$ (7)

where Y_t, and r_0 are $n$vectors and $R$is a real $n \times n$square matrix. A class of R matrices are established where R defines restrictions on coefficients of (7).

In the Journal of Czechoslovak Mathematical (1985), GaryfalosPapashinopoulos, John Schinas, Xanthistudied the difference equation with criteria for an exponential dichotomy of the form,

$y(n+1) = A(n)y(n)$ (8)

where$A(n)$ is$k \times \mathrm{k}$ invertible matrix. The necessary and sufficient conditions for the solution of equation (8) are obtained.

In the Journal ofComputers and Mathematics with Applications (1995),M.P.Chen, B.S.Lalli and J.S.Yuestablishedthe results for neutral delay difference equationof the form,

$\Delta(y_n - p_n y_{n-k}) + q_n y_{n-l} = 0, \quad n = 0,1,2, \ldots$ (9)

The aim is to establishthe sufficient conditions for oscillation solutions of (9) neglecting thecondition,

$\sum_{n=0}^{\infty} q_n = \infty$ (10)

In the Journal of Mathematical and Computer Modelling (1998), B.Szmandastudiedthe solutions and established someproperties of the equation given as,

$\Delta^n [u(x) + p(x)u(x-k)] = q(x)f\left(u(\tau(x))\right), \quad n \geq 1, x \in N$ (11)

where $N = \{0,1,2, \ldots\}$, $k = \{1,2,3, \ldots\}$, $f: R \to R$, $uf(u) > 0, u \neq 0$. The asymptotic behaviour of non-oscillatory solutions for (11) isstudied.

In the Journal of Computers and Mathematics with Applications (2001), M.Pitukstudied thelimits of solutions with initial conditions. The non-autonomous linear equation takes the form,

$x_{p+1} - x_p = \sum_{j=1}^{m} A_j(p)\left(x_{p-k_j}, x_{p-l_j}\right), \quad p = 0,1,2, \ldots$ (12)

where $m \geq 1$, the coefficient $A_j(p)$ is a square matrix and k_j, l_j are non-negative integers. Solution of (12) tends to a constant when $p \to \infty$.

In the Journal of Applied Mathematics Letters (2002), Yuming Chen discussed the solutions of difference equations given as,

$x_{n+1} = x_n - f(x_{n-k}), \; n = 0,1, \ldots$ (13)

where$k$ is an integer. The solutions of (13) are truncated periodic. The established results have their application in neural networks.

In 2003, the Journal of Applied Mathematics and Computation shows the work ofH.El-Owaidy and H.Y.Mohamed whostudied the periodic solutions of difference equations with necessary and sufficient conditions. The first and second order equations are given as,

$y_{m+1} = F(m, y_m)$ (14)

$y_{m+1} = F(m, y_m, y_{m-1})$ (15)

where$m = 0,1,2, \ldots$The rational difference equation is

$y_{m+1} = \frac{(a_m + b_m y_m)}{(y_{m-1})}$ (16)

The results of (14) and (15) are applied to (16).

In the Journal of Computers and Mathematics with Applications (2005), Xiaochun Cai, Jianshe Yu and ZhumingGuo considered the equation of the form,

$\Delta^2(r_{n-2}\Delta^2 x_{n-2}) + f(n, x_n) = 0$ (17)

where $n \in Z$. The periodic solutions and condition of existence are obtained.

In 2007 the Journal of *Advances in Difference Equations* shows the work done byXiaochunCai and Jianshe. The existence theorems areestablishedfor periodic solutions of2^{nd} order difference equation

$$\Delta\big(p_n(\Delta x_{n-1})^{\delta}\big) + q_n x_n^{\delta} = f(n, x_n),\ n \in \mathbb{Z} \qquad (18)$$

where$\{p_n\}$ and$\{q_n\}$ are real numbers and$\delta$is the ratio of integers.

In the Journal of Discrete Dynamics in Nature and Society (2010), Bratislav D. Iričanin and Wanping Liu studiedthe difference equationof higher order given as,

$$y_m = c y_{m-p} y_{m-p-q} / y_{m-q} \qquad (19)$$

where $p, q \in \mathbb{N}, c > 0$. The behaviour of positive solutionis studied.

In the Journal of Applied Mathematics and Computation (2012), Stevo Stevic discussed the difference equation for third order system,

$$x_{n+1} = \frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2} + c_1},\ y_{n+1} = \frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2} + c_2},\ z_{n+1} = \frac{a_3 z_{n-2}}{b_1 x_n y_{n-1} z_{n-2} + c_3} \qquad (20)$$

where$a_i, b_i, c_i\, \&\ i \in \{1,2,3\}$are parameters and$x_{-j}, y_{-j}, z_{-j},\ j \in \{0,1,2\}$are the initial values. Riccati's system is used for reduction method of (20). Results are solved for the following cases:

1. When $a_i = 0$ for some $i \in \{1,2,3\}$
2. When $a_i \neq 0$ for all $i \in \{1,2,3\}$
3. When initial values equals zero

The main result of (20) is solved when the above three cases fail.

In 2013 the Journal of *Advances in Difference Equations* gives the work ofMariasebastin Maria Susai Manuel, Adem Kılıçman, Gnanadhass Britto Antony Xavier, RajanPugalaras and DevadanamSuseelaDilip. An application is establishedfor2^{nd} order difference equation of the form,

$$\Delta_l^2 u(n) + f\big(n, u(n)\big) = 0 \qquad (21)$$

where $n \in [a, \infty)$. Solution for (21) is discussed.

In 2015 the Journal of Difference Equations shows the work of Ali Gelisken and MerveKara,who considered the rational difference equations of general systemgiven as,

$$y_{m+1} = z_{m-(3k-1)} / \pm 1 \pm z_{m-(3k-1)} y_{m-(2k-1)} z_{m-(k-1)} \qquad (22)$$

$$z_{m+1} = y_{m-(3k-1)} / \pm 1 \pm y_{m-(3k-1)} z_{m-(2k-1)} y_{m-(k-1)} \qquad (23)$$

where$k$is a positive integer, $m = 0,1,2, ...$The cases for$k = 1,\ k = 2$ are given for (22) and (23). The solution is said to be periodic withperiod $6k$.

In the Journal of Mathematics (2018), Melih Gocen and AdemCebecistudied the solutions of difference equations with higher order. Thesystem of equations are given as,

$$x_{n+1} = \frac{\pm x_{n-k} y_{n-(2k+1)}}{y_{n-(2k+1)} \mp y_{n-k}} \quad \text{and} \quad y_{n+1} = \frac{\pm y_{n-k} x_{n-(2k+1)}}{x_{n-(2k+1)} \mp x_{n-k}} \qquad (24)$$

where $n, k \in \mathbb{N}_0$. The general form of the periodic solutions is obtained. Numerical examples are presented.

In the ofJournal of Applied Analysis and Computation (2019), Shugui Kang, Huiqin Chen, Luping Li, Yaqiong Cui and Shiwang Ma discussed the existence of solutions for Riemann-Liouville class of fractional Q-difference equationgiven by,

$$\big(D_q^{\alpha} x\big)(s) + f\big(s, x(s)\big) = 0,\ 0 < q < 1, 0 < s < 1 \qquad (25)$$

whereD_q^{α} isthe fractional q-derivative. Authors employed the existence of solutionusing fixed point theorem proposed by Leggett-Williams.

1.2 Survey Results on Oscillation of Difference Equations

A solution $x(n)$ for a difference equation iscalledoscillatoryif the terms $x(n)$are neither eventually positive nor eventually negative. If not, the solution is said to benon-oscillatory.

The Journal of Society for Industrial and Applied Mathematics (1979), provides the work of William T. Patulawho studied the second order linear difference equations with growth and oscillation propertiesgiven as,

$c_n x_{n+1} + c_{n-1}x_{n-1} = b_n x_n,\ c_n > 0$ (26)

where $b_n \geqq 0$. The properties of solution are investigated. Results of limit point are obtained and sufficient conditions for oscillation are presented.

In the Journal of Annales Polonici Mathematici (1983), BfazejSzmanda discussed the criteria of oscillation for 2^{nd} order difference equationgiven as,

$\Delta(r_n \Delta u_n) + a_n f(u_n) = 0,\ n = 0,1,\ldots$ (27)

Where$\{a_n\}$and $\{r_n\}$ are real number sequences. Sufficient conditions of oscillation are establishedfor the equation.

In the Journal of Demonstratio Mathematica (1984), Jerzy Popenda and BfazejSzmandaconsideredthe difference equation of the form,

$\Delta_a x_n + \delta \sum_{i=1}^{m} q_{in} f_i(x_{d_{in}}) = 0,\ n = 0,1,2,\ldots$ (28)

where $\delta = \pm 1$, $\{q_{in}\}, i = 1,\ldots,m$ are real number sequence and $\{d_{in}\}, i = 1,2,\ldots,m$. The oscillation solutions for(28) are established.

In the International Journal ofMathematics and Mathematical Sciences (1990), B.Smith and W.E.Taylorstudiedthe non-oscillatory and oscillatory behaviour for difference equation given as,

$\Delta(p_n \Delta^2 v_n) + Q_n f(v_{n+1}, \Delta v_{n+1} + \Delta^2 v_{n+1}) = 0,\ n = 1,2,\ldots$ (29)

The criteria of oscillation aredetermined for second order equations.

In the Journal of Colloquium Mathematicum(1993), B.S.Lalli studied the difference equation of oscillation. The equation is of the form,

$y_{n+1} - y_n + \sum\ p_{in} y_n - m_i = 0$ (30)

where $m_i, i = 1,2,\ldots,K$, the positive integers and $n = 1,2,\ldots$ A solution $\{y_n\}$ is considered and the oscillatory criteria are investigated.

In the Journal ofComputers and Mathematics with Applications (1996), E. Thandapani, P. Sundaram and B. S. Lalli established the theorems on oscillationof higher order nonlinear difference equations. The higher order difference equation and forced difference equation are given by,

$\Delta^m y(n) + q(n) f\left(y(\sigma(n))\right) h\left(\Delta^{m-1} y(\delta(n))\right) = 0,\quad n = 0,1,2,\ldots,\mathrm{m}$(31)

$\Delta^m y(n) + q(n) f\left(y(\sigma(n))\right) h\left(\Delta^{m-1} y(\delta(n))\right) = e(n),\ n = 0,1,2,\ldots,\mathrm{m}$ (32)

The oscillation results forms discrete analogues of (31) and (32).Examples are provided.

In the Journal of Computers and Mathematics with Applications (1998), B.G. Zhang, Jun Yan and S.K. Choi studied the oscillation condition with continuous variable for the nonlinear difference equationof the form,

$x(s) - x(s-\tau) + q(t) H\left(x(s-\sigma)\right) = 0,\ s \geq 0,$ (33)

where $\tau > 0$ and $\sigma > 0$. Sufficient conditionsofoscillationfor forced difference equation are established.

In the Journal of Applied Mathematics Letters (2000), X.H. Tang and J.S. Yu studied the delay difference equationsof oscillationwith a critical state. The equation is of the form,

$\Delta x_n + p_n x_{n-k} = 0,\ n = 0,1,2,\ldots$ (34)

where$\{p_n\}$is the non-negativesequence and k is an integer. The oscillationcondition,$\liminf_{n\to\infty} p_n = \frac{k^k}{(k+1)^{k+1}}$for (34) is established.

In the Journal of Applied Mathematics Letters (2003), Wan-Tong Li and Sui-Sun Cheng considered thelinear difference equation and established the oscillation theorems for the following equation

$\Delta^2 x_{n-1} + p_n x_n = 0$ (35)

where $\{p_n\}$ is a non-negative sequence.

In the Journal ofSoochow Journal of Mathematics (2005),Ravi P. Agarwal, Said R. Graceand DonalO'Regandiscussed the oscillatory criteria of higher order equation given by

$\Delta(\Delta^{m-1}x(n))^{\alpha} + q(n)x^{\alpha}[n-\tau] = 0$ (36)

where $m \geq 2,\ \tau \geq 1$ and α is the ratio of odd positive integers. The extensions of solutions are investigated.

In the Journal of Computers and Mathematics with Applications (2006), Xiaoyan Lin considered the neutral difference equations and established theoscillation solutionswith a neutral term. The equation is,

$\Delta(x_n - p_n x_{n-\tau}^{\alpha}) + q_n x_{n-\sigma}^{\beta} = 0, \qquad n \geq n_0$ (37)

where $\alpha,\ \beta$ and τ are positive integers, σ is non-negative integer,$\{p_n\}$ and $\{q_n\}$ are real number sequences.The conditions$\alpha \neq 1$ and $0 < \alpha < 1$ for (37) is studied. Conditionfor oscillatory and non-oscillatory is obtained when$\alpha > 1$.

In the Journal of *Advances in Difference Equations* (2006), Yinggao Zhouestablishedthe oscillation condition for delay difference equation of theform,

$\Delta^{l} x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} = 0, \quad n = 0,1,2,...$ (38)

where the sequence $\{p_i(n)\}$ are non-negative real numbers, k_i is a positive integer. The corresponding inequality of first order is given as,

$\Delta\ \ x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} \leq 0, \quad n = 0,1,2,...$ (39)

Sufficient conditions of oscillation are obtained.

In the Journal of Applied Mathematics Letters (2007), M.K. Yildiz and O. Ocalanstudied the neutral delay difference equations of higher order and established the oscillation results. The nonlinear equation is of the form,

$\Delta^{m}(y_n + p_n y_{n-l}) + q_n y_{n-k}^{\alpha} = 0$ (40)

where$\{p_n\}$ and $\{q_n\}$ are real number sequences,$\alpha \in (0,1)$ is ratio of positive integers. The oscillatory result for (40) is obtained using the conditions as follows:

(1) $0 \leq p_n < 1$

(2)$0 \leq p_n \leq P_1 < 1,\ P_1$is constant

(3) $-1 < -P_2 \leq p_n \leq 0,\ P_2 > 0$is constant

(4)$\sum_{n=0}^{\infty} \quad q_n[(1-p_{n-k})(n-k)^{m-1}]^{\alpha} = \infty$

(5)$\sum_{n=0}^{\infty} \quad q_n[(n-k)^{m-1}]^{\alpha} = \infty$

In the Journal of ComputersandMathematicswithApplications (2009), HadiKhatibzadehstudied thedifference equationgiven as,

$x_{n+1} - x_n + p_n x_{n-k} = 0, \qquad n = 0,1,2,...$ (41)

where$\{x_n\}$isthe real number sequence and the solution of (41) is said to be oscillatory. Using the condition proposed byYu, Zhang and Qian, the equationis of the form,

$\sum_{i=n-k}^{n} p_i \quad < 1, \quad n = 0,1,2,...$ (42)

the oscillation criteria for (41) are established.

In the Journal of ComputationalandAppliedMathematics (2010), WeiLu, WeigaoGe and ZhihongZhaodiscussed the 3^{rd} orderdifferenceequation and established the oscillation criteriawithimpulses. The equation takes the form,

$$\begin{cases} \Delta^3 x(n) + p(n) f\big(x(n-\tau)\big) = 0, & n \neq n_k, \quad k = 1,2,3,... \\ \Delta^i x(n_k) = g_{i,k}(\Delta^i x(n_k - 1)), & i = 0,1,2,\ k = 1,2,3,... \end{cases} \qquad (43)$$

where$p(n) \geq 0$.The oscillatory criteria are obtained with the following conditions:

1) $f: \mathbb{R} \to \mathbb{R}$ is continuous and $f(u)/u \geq \varepsilon_0 > 0$ with $u \neq 0$and $\varepsilon > 0$.

2) $g_{i,k}(u)$ is continuous and $a_{i,k},\ b_{i,k}$ are positive numbers for$a_{i,k} \leq \frac{g_{i,k}(u)}{u} \leq b_{i,k},\ u \neq 0,\ i = 0,1,2,$and $k = 1,2,3,...$Examples are provided.

In the Journal of Applied Mathematics & Information Sciences (2012), Yasar Bolat and Jehad O. Alzabut studied the delay difference equation of oscillation for the form,

ISSN- 2394-5125 VOL 7, ISSUE 11, 2020

$$\Delta(p_n(\Delta^{m-1}(x_n + q_n x_{\tau_n}))^{\alpha}) + r_n x_{\sigma_n}^{\beta} = 0, \ n \geq n_0 \qquad (44)$$

where $\Delta p_n \geq 0$. The oscillation and asymptotic criteria for (44) are established with necessary and sufficient conditions. Examples are presented.

In the Journal of Advances in Difference Equations (2015), Chunhua Yuan, Shutang Liu and Jian Liu studied the partial difference equation with oscillation. Consider the equation

$$u_{n+2,m} + u_{n,m+2} + au_{n+1,m} + bu_{n,m+1} + cu_{n,m} = 0 \qquad (45)$$

where$a, b, c$are the real numbers and $n, m$are the integers. The necessary and sufficient conditionsare developedby theory of envelopes for solutions of oscillation.

In Electronic Journal of Differential Equations (2017), Srinivasan Selvarangam, Mayakrishnan Madhan, Ethiraju Thandapani and Sandra Pinelasestablished the oscillationcondition forneutral typeequation given as,

$$\Delta\left(a_n\left(\Delta^2(x_n + p_n x_{n-k})\right)^{\alpha}\right) + q_n f(x_{n-l}) = 0 \qquad (46)$$

where $\alpha > 0$, $a_n > 0$, $q_n \geq 0$, $0 \leq p_n \leq p < \infty$. With Ricatti type transformationall solutions oscillate. Examples are provided.

In the Journal of Advances in Difference Equations (2018), M. Nazreen Banu and S. Mehar Banu studied the second order quasi-linear difference equationsand established oscillation conditionwith neutral term. The equation is,

$$\Delta\left(a_n(\Delta z_n)^{\beta}\right) + q_n x_{n-l}^{\gamma} = 0, \ n \geq n_0 \qquad (47)$$

where $\mathbb{N}(n_0) = \{n_0, n_{0+1}, \dots\}$, n_0 being non-negative integer. The necessary and sufficient conditions are obtained. Examples are presented.

In 2019the Journal of Advances in Difference Equations (2019) shows the work of S. Kaleeswariwho discussed the higher order nonlinear neutral difference equation and investigated the oscillation criteriafor the following form

$$\Delta^m\left(x(n) + p(n)x(\tau(n))\right) + q(n)f\left(x(\sigma(n))\right) = 0, \ n \in N = \{0,1,2,\dots\} (48)$$

where$m \geq 1$. Examples are provided.

In the International Journal of Engineering and Advanced Technology (2019), S. Sindhuja, J. Daphy Louis Lovenia, A.P. Lavanya and Gomathi Jawahar studied the qualitative analysis of linear difference equationsforfirst ordergiven as,

$$\nabla y(t) - p(t)y(\sigma(y)) = 0, \quad t \in \mathbb{N} \qquad (49)$$

where$\mathbb{N}$denote the positive integers, $\{p(t)\}$ represents the real number sequence and $\{\sigma(t)\}$ represents the integer sequence. Sufficient condition of oscillation is established for (49) that involvesinf. Iterative method is used and examples areprovided.

1.3 Survey Results on Stability and Asymptotic Behaviourof Difference Equations

The equilibrium point x^* of the difference equation,

$$x(n+1) = f(x(n)) \qquad (50)$$

is stable if $\varepsilon > 0$ there exist $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If not then$x^*$is unstable. The point x^* is asymptotically stable if the equilibrium point is stable and attracting. If $\eta = \infty$, then x^* is globally asymptotically stable.

In the Journal für die reine und angewandte Mathematik (1966),W. A. Harris, JR., and Y. Sibuya studiedthe system of nonlinear difference equationsand investigated the asymptotic solutionsfor the equation given as,

$$x(y+1) = f(y, x) \qquad (51)$$

where$y$is a complex variable and $y$is a vector. The existence of analytic solution for (51) is shown.

In the Journal of Applied Mechanics (1977), C. S. Hsu,H. C. Yee and W. H. Chengdiscussedthe asymptotic stability for determiningglobal regions of the dynamical systemgiven as,

$x_i(s+1) = f_i\big(x_1(s), x_2(s), \dots x_N(s)\big), \qquad i = 1,2,\dots,N$ (52)

where f_i is a real function. The stability for the system is established.

In 1986the Journal of International Mathematics and Mathematical Sciences shows the work of B.Smithwho studied theasymptotic and oscillatory behaviourof quasi-adjoint3^{rd}order equationof the form,

$\Delta^3 v_n + p_{n-1} v_{n+1} = 0$ (53)

where $\{v_n\}$ and $\{p_n\}$ are sequences of real numbers. The quasi-adjoint equation is given as,

$\Delta^3 u_n + p_n u_{n+2} = 0$ (54)

(53) is investigated by (54). The necessary and sufficient conditionsare given. Example for non-oscillatory solution is shown.

In the Journal of Applicable Analysis (1990), V. Lakshmikantham studied the stability solutions for linear difference equationgiven as,

$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} = 0, \ n = 0,1,2,\dots$ (55)

where $k_i \in \mathbb{N}, p_i \in \mathbb{R}$ for $i = 1,2,\dots,m$. The asymptotic behaviour is investigated and conditionsfor global asymptotic stability are established.

In 1993 the Journal ofFundamental Theory and Applications provides the work of P. Bauer, M. Mansour, and J. Duran where the stability is established for polynomialswith time-varying coefficients. The m^{th} order difference equation is given as,

$z(x) = a_1(x)y(x-1) + \cdots + a_n(x)y(x-n)$ (56)

The following condition is considered

$\sum_{\nu=1}^{n} a_\nu^+ < 1$ (57)

where $|a_\nu(x)| \le a_\nu{}^+$ for $\nu = 1,2,\dots,n$, (56) becomesasymptotically stable. Results are derived for the cases oftime-variant and time-invariant regions.

In the Journal of International Mathematics and Mathematical Sciences (1994), John R. Graef and Paul W. Spikes discussed thesolutions for forced difference equation with boundedness and asymptotic behaviour. The equation is,

$\Delta[y_n + p_n y_{n-h}] + q_n f(y_{n-k}) = r_n$ (58)

where $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$are real number sequences, $k \in \mathbb{N} = \{0,1,2,\dots\}$. Sufficient conditionsare given and examples are provided.

In the Journal of Computational and Applied Mathematics (1997), AndrzejDrozdowicz and MatgorzataMigdaconsidered the solutions of asymptotic behaviourforn^{th}order equation taking the form,

$\Delta^n x_m + f(m, x_m, \dots, \Delta^{n-1} x_m) = h_m$ (59)

where $h: \mathbb{N} \to \mathbb{R}$, $f: \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, $n \ge 2$.The asymptotic behaviour of non- homogeneous second order equation of the following form is investigated.

$\Delta^2 x_m + g_0(m) x_m^{t_0} + g_1(m)[\Delta x_m]^{t_1} = h_m$ (60)

where$h: \mathbb{N} \to \mathbb{R}$ and (60) is a special case of (59). The asymptotic behaviour is established for solutionx_m.

In the Journal of Computers and Mathematics with Applications (1998),Binxiang Dai and Lihong Huangconsidereda class of nonlinear equationsand studied the solutions of asymptotic behaviour for equation given as,

$x_{n+1} - x_n = F(n, x_n, x_{n-k}), \ n = n_0 + 1, n_0 + 2, \dots$ (61)

where k is an integer. The solution of (61) converges to a constant.

In the Journal of Difference Equation and Applications (1999), George Karakostas discussed the2-periodasymptoticdifference equation with self-invertible responses. The equation is,

$x_{n+1} = f(x_n, x_{n-2}), \ n = 0,1,2,\dots$ (62)

Sufficient conditionsare establishedfor the third order equation to be asymptotic.

In the Journal of Applied Mathematics Letters (2000), B. G. Zhang discussed thebehaviour of solutionforthe equations given as,

$$\Delta y_m = \sum_{l=1}^{t} b_l(m) f_l(y_{\tau_l(m)}) = 0,\ m \geq 0 \qquad (63)$$

and

$$\Delta y_m = a_m y_m + \sum_{l=1}^{t} b_l(m) \left|y_{\tau_l(m)}\right|^{\gamma l} sgn y_{\tau_k(m)} = 0,\ m \geq 0 \ (64)$$

The asymptotic behaviour of (63) and (64) are obtained.

In the Journal of Applied Mathematics Letters (2002), E.Liz and J. B. Ferrerioestablishedthe global stabilityofgeneralized equation of the form,

$$\Delta x_n = -a x_n + f(n, x_n, x_{n-1}, \dots, x_{n-r}),\ a > 0 \qquad (65)$$

where $n = -r, \dots, 0$. Halanay inequality is considered and a sufficient condition is derived.

In the Journal of *Advances in Difference Equations* (2004) CH. G. Philos and I. K. Purnaras considered the equationwithasymptotic result having continuous variable given as,

$$x(s) - x(s-\sigma) = a x(s-\sigma) + \sum_{j=1}^{k} b_j x\left(s - \tau_j\right) + f(s) \qquad (66)$$

where k being a positive integer, a and $b_j \neq 0 (j = 0,1,2, \dots, k)$ are constants.

In the Journal of Mathematics and Mathematical Sciences (2005), T. Kaewong, Y.Lenbury and P. Niamsupestablishedthe stability conditions for equation given by,

$$y_{m+1} - y_m + p \sum_{j=1}^{m} y_{m-k} + (j-1)l = 0 \qquad (67)$$

where$p$ is a real number, k, l, N being the positive integers and $k > (N-1)l$. With the following condition the necessary and sufficientconditions areestablished for (67)

$$y_{m+1} + p \sum_{j=1}^{m} y_{m-k} + (j-1)l = 0 \qquad (68)$$

where (68) is independent of y_m.

In the Journal ofDiscrete Dynamics in Nature and Society (2008), Fangkuan Sundiscussed the behaviour of equation given as,

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\},\ \text{n} = 0,1,2, \dots \qquad (69)$$

where $0 < \alpha,\ \beta < 1,\ A, B > 0$. Also, the solution of (70) converges tox^*, where

$$x^* = max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\} \qquad (70)$$

In the Journal of Discrete Dynamics in Nature and Society (2009)**,** Meseret Tuba Gulpinar and Mustafa Bayramconsidered the4^{th}order difference equation and studied thestability of equation of the form,

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-3} + x_n + x_{n-2} + x_{n-3} + a}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a},\ n = 0,1,2, \dots \qquad (71)$$

where $x_{-3},\ x_{-2}, x_{-1}, x_0 \in (0, \infty)$ are initial values,$a \in (0, \infty)$. The equilibrium point for (71) is studied using positive and negative semi-cycles.

In the Journal of MathematicalandComputerModelling (2011), MehdiDehghan andNargesRastegarestablished the stabilityforequation given as,

$$y_{n+1} = \frac{\alpha y_{n-2}}{\beta + \gamma y_n^k y_{n-1}^k y_{n-2}^k},\ n = 0,1,2, \dots \qquad (72)$$

where y_{n-2}, y_{n-1}, y_n are initial conditions, α, β, γ are positive real numbers and$k \geq 2$ is a fixed integer. Stability, periodic character and boundedness of solution are investigated.

In the Journal of Applied Mathematics Letters (2012), Qi Wang, Fanping Zeng, Xinhe Liu, Weiling Youconsideredthe stability for a rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, n = 0,1,2, \dots \qquad (73)$$

where $\alpha, \beta, \gamma, A, B, C \in (0, \infty)$. Conjectures proposed by Kulenović and Ladas are investigated. Linearized stability theorem is established and numerical results are obtained.

In the Journal of Applied Mathematics and Computation (2014), Lin-Xia Hu and Hong-Ming Xia discussed the 2^{nd} order equation with stability given as,

$$y_{n+1} = \frac{p_n + y_n}{p_n + y_{n+1}}, \quad n = 0,1,2, \dots \tag{74}$$

where $p_n = \begin{cases} \alpha, \text{if } n \text{ is even} \\ \beta, \text{if } n \text{ is odd} \end{cases}$ and $\alpha > 0, \beta > 0, \alpha \neq \beta$.

An open problem for global character of the sequence $\{p_n\}_{n=0}^{\infty}$ with period two is investigated.

In the Journal ofAppliedMathematicsLetters (2015), JanČermák, JiříJánskýstudied the linearequationwith conditions of explicit stabilitygiven by,

$$x(m) = \alpha x(m-n) + \beta x(m-k) \tag{75}$$

whereα and β are real constant coefficients, $k > n > 0$ are integers. The stability criteriafor third type are formulated.

In the Journalof Linear Algebra and its Applications (2016), Nguyen H. Sau, P. Niamsup and Vu N. Phat considered the linear difference equations and analysed stability of equation given by,

$$\begin{cases} Ex(p+1) = A_0 x(p) + A_1 x(p - h(p)), \ p \in \mathbb{N}, \\ \quad x(p) = \varphi(p), \ p \in \{-\tau, -(\tau-1), \dots 0\}, \end{cases} \tag{76}$$

where $x(p) \in R^n$, $A_0, A_1 \in R^{n\times n}$ and $E \in R^{n\times n}$ with rank $E = r < n$. With mathematical induction and decomposition method, the necessary and sufficient conditions are derived. Examples are provided.

In the Journalof International Federation of Automatic Control (2017), Qian Ma, Keqin Gu and NargesChoubedar studied the stability of equations with continuous time given as,

$$y_k(t) = \sum_{j=1}^{k} d_{kj}\, y_j(t - \tau_j), \ k = 1,2, \dots, K \tag{77}$$

where $y_k \in \mathbb{R}$, $d_{kj} \in \mathbb{R}$, $k, j = 1,2, \dots, K$. The ODPSC- one delay per scalar channel model is set to (77). The equivalence relation is shown between strong stability, exponential stability and stability of delays.

In the of Journal of Applied Mathematics Letters (2018), Guo-Cheng Wu, Dumitru Baleanu and Lan-Lan Huang discussedthe Mittag-Leffler stability of fractional difference equations given by,

$$^{C}D_a^{\alpha} u(t) = \lambda u(t), \ 0 < \alpha \leq 1, t \geq a, u(a) = u_0 \tag{78}$$

where $^{C}D_a^{\alpha} u(t)$is Caputo fractional derivative.The comparison principle and stability results are used. With mathematical induction the results are derived. Examplesare established.

In the Journal of Automatica (2019), BinZhoudiscussed the robust and strongstability of linear equation withdelays taking the form,

$$x(t) = A_1 x(t - r_1) + A_2 x(t - r_2) + \cdots A_N x(t - r_N) \tag{79}$$

where$r_i > 0, i = 1,2, \dots, N$ and $A_i$is the square matrix. A special case of (79) is investigated. The linear equation is,

$$x(t) = Ax(t - a) + Bx(t - b) \tag{80}$$

where the constants $a > 0$, $b > 0$ and for A, B the square matrices are given as $n \times n$. Examples are provided.

CONCLUSION

This paper provides thesurvey results on difference equations under three categories. The collected articles deploy a theoreticalway to approach the concepts. The overall work givesa well-defined presentation in an understandable way

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