A TRIPLE FIXED POINT THEOREM OF CARISTI TYPE CONTRACTION FOR MULTI VALUED MAPS IN A HAUSDORFF METRIC SPACE

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Abstract
The main aim of this paper is to obtain a unique common tripled fixed point of caristi type caristi type ontraction for multi valued mappings in a Hausdorff metric space.

Keywords: Metric space, compatible maps, tripled fixed point, Hausdorff metric.

INTRODUCTION
The concept of standard metric space is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric space. Since Banach introduced this theory in 1922([10]), it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modified ε-variation principle of Ekeland([9]). In 1976, Caristi proved the following famous fixed point theorem.

Theorem 1.1 [6] Let (X, d) be complete metric space and f : X → R be lower semi continuous function and bounded below function. A mapping T : X → X is said to be Caristi type map on X dominated by f if T satisfies d(x,Tx) ≤ f(x) − f(Tx) for each x ∈ X. Then T has a fixed point.

S.B.Nadler introduced the concept of multivalued contraction mappings in the year 1969([11]).

Definition 1.2 ([11]) Let (X, d) be a metric space. We define the Hausdorff metric on CB(X) induced by d. That is H(A, B) = max{sup d(l, B), sup d(m, A)} for all A, B ∈ CB(X), where CB(X) denotes the family of all nonempty closed and bounded subsets of X and d(l, B) = sup {d(l, b); b ∈ B}, for all l ∈ X.

Definition 1.3 ([11]) Let (X, d) be a metric space. A map T : X → CB(X) is said to be multivalued contraction if there exists 0 ≤ α < 1 such that H(T(l), T(m)) ≤ αd(l, m), for all l, m ∈ X.

Lemma 1.4 ([8]) Let X be a nonempty set and g : X → X be a mapping, then there exists a subset E ⊆ X such that g(E) = g(X) and g : E → E is one one.

Now we give the following definitions for hybrid pair of mappings.

Definition 1.5 ([7]) Let X be a non empty set, T : X × X × X → 2^X(collection of non empty subsets of X) and f : X → X.

(i) The point (l, m, n) ∈ X × X × X is called a tripled fixed point of T if

\[ l \in T(l, m, n), \quad m \in T(m, l, m), \quad n \in T(n, m, n). \]

(ii) The point (l, m, n) ∈ X × X × X is called a tripled coincident point of T and f if

\[ f l \in T(l, m, n), \quad f m \in T(m, l, m), \quad f n \in T(n, m, n). \]

(iii) The point (l, m, n) ∈ X × X × X is called a tripled common fixed point of T and f if

\[ l = f l \in T(l, m, n), \quad m = f m \in T(m, l, m), \quad n = f n \in T(n, m, n). \]

Definition 1.6 [7] Let T : X × X × X → X be a multi valued map and f be self map on X. The hybrid pair (T, f) is called w−compatible if T(f(l, m, n)) ⊆ T(f(l, m, n)) whenever (l, m, n) is tripled coincidence point of T and f.

Lemma 1.7 [See [5]] Let $\alpha$ be a reflexive relation on a nonempty set $M$ and $\phi : M → R$ a function bounded from below, then $\alpha = \phi$ and $\phi = \phi(y)$.

Throughout this paper, we assume that $\zeta : [0, \infty) → [0, \infty)$ is an upper semi continuous function.

Now we prove our main results.

RESULTS AND DISCUSSIONS
Theorem 2.1 Let (X, d) be a complete metric space and let S : X × X × X → CB(X) be a set valued mapping satisfies

\[ H(S(l, m, n), S(a_1, b_1, c_1)) \leq \max \left\{ \begin{array}{l} \zeta \left( \max \{\zeta(l, a_1), \zeta(m, b_1), \zeta(n, c_1)\} \right), \\
\zeta \left( \max \{\zeta(a_1, b_1), \zeta(a_2, b_2), \zeta(a_3, b_3)\} \right) \\
\max \{\zeta(l, a_1), \zeta(m, b_1), \zeta(n, c_1)\} \\
\max \{\zeta(a_1, b_1), \zeta(a_2, b_2), \zeta(a_3, b_3)\} \end{array} \right\} \]

where $\zeta : [0, \infty) → [0, \infty)$ is a generalized contraction coefficient.
for some $a_j \in S(l, m, n), a_{j+1} \in S(m, l, m_1), a_{j+2} \in S(n, m, I)$, $\beta_j \in S(b_j, b_{j+1}), \beta_{j+1} \in S(b_{j+1}, c_{j+1}), \beta_{j+2} \in S(c_{j+1}, b_{j+2})$. Where $S: \mathbb{X} \times \mathbb{X} \times \mathbb{X}$ is lower semi continuous function and $\zeta: [0, \infty) \to [0, \infty)$ be an upper semi continuous function. Then $S$ has a triple fixed point.

**Proof:** Define a relation $\sim$ on $X$ as follows:

$$S(l, m, n) \sim S(a_j, b_j, c_j) \iff H(S(l, m, n), S(a_j, b_j, c_j)) \leq \max \left\{ \zeta \left( \max \left( \zeta \left( (l, a_j), (m, b_j), (n, c_j) \right) \right), \zeta \left( \max \left( \zeta \left( (l, a_j), (m, b_j), (n, c_j) \right) \right) \right) \right\}$$

Then clearly $\sim$ is a reflexive relation on $X$.

Let $l_0, m_0, n_0 \in X$ be arbitrary points in $X$.

Choose $l_0 \in S(m_0, n_0, m_0, m_0, n_0, m_0, n_0, l_0)$ since $S$ is compact valued maps so there exists $l_0 \in S(m_1, m_1, m_1)$ and $m_0 \in S(l_0, m_0, l_0)$ such that $d(l_1, l_2) \leq H(S(l_0, m_0, n_0), S(l_0, m_0, n_0))$

$$\leq \max \left\{ \zeta \left( \max \left( \zeta (l_1, l_2), (m_0, m_0), (n_0, n_0) \right) \right), \zeta \left( \max \left( \zeta (l_1, l_2), (m_0, m_0), (n_0, n_0) \right) \right) \right\}$$

and

$$d(m_0, m_0) \leq H(S(m_0, l_0, m_0, n_0), S(m_1, m_1, m_1)) \leq \max \left\{ \zeta \left( \max \left( \zeta (m_0, m_0), (l_0, l_1), (m_0, m_0) \right) \right), \zeta \left( \max \left( \zeta (m_0, m_0), (l_0, l_1), (m_0, m_0) \right) \right) \right\}$$

Also

$$d(n_0, n_0) \leq H(S(m_0, n_0, m_0, n_0), S(n_0, m_0, n_0)) \leq \max \left\{ \zeta \left( \max \left( \zeta (n_0, n_0), (l_0, l_1), (n_0, n_0) \right) \right), \zeta \left( \max \left( \zeta (n_0, n_0), (l_0, l_1), (n_0, n_0) \right) \right) \right\}$$

Therefore

$$\max (d(l_1, l_2), d(m_0, m_0), d(n_0, n_0)) \leq \max \left\{ \zeta \left( \max \left( \zeta (l_1, l_2), (m_0, m_0), (n_0, n_0) \right) \right), \zeta \left( \max \left( \zeta (l_1, l_2), (m_0, m_0), (n_0, n_0) \right) \right) \right\}$$

Continuing in this way we can obtain sequences $(l_k, m_k, n_k)$ in $X$ such that $l_{k+1} \in S(l_k, m_k, n_k), m_{k+1} \in S(m_k, l_k, m_k)$ and $n_{k+1} \in S(n_k, m_k, l_k)$ such that

$$\max (d(l_k, l_{k+1}), d(m_k, m_{k+1}), d(n_k, n_{k+1})) \leq \max \left\{ \zeta \left( \max \left( \zeta (l_k, l_{k+1}), (m_k, m_{k+1}), (n_k, n_{k+1}) \right) \right), \zeta \left( \max \left( \zeta (l_k, l_{k+1}), (m_k, m_{k+1}), (n_k, n_{k+1}) \right) \right) \right\}$$
\[ \leq \max \left\{ \zeta \left( \max \{ \zeta(l, m), \zeta(m, n) \} \right), \right. \\
\left. \zeta \left( \max \{ \zeta(a, b), \zeta(a, b), \zeta(a, b) \} \right) \right\} \\
\left\{ \max \{ \zeta(l, m), \zeta(m, n) \} \right\}\right].
\]

Letting \( k \to \infty \) we have

\[ H(S(l, m, n), S(l, m, n)) \leq \max \left\{ \zeta \left( \max \{ \zeta(l, m), \zeta(m, n) \} \right), \right. \\
\left. \zeta \left( \max \{ \zeta(a, b), \zeta(a, b), \zeta(a, b) \} \right) \right\} \\
\left\{ \max \{ \zeta(l, m), \zeta(m, n) \} \right\}\right] = 0.
\]

Therefore \( H(S(l, m, n), S(l, m, n)) = 0 \).

Similarly we can prove that

\[ H(S(m, l, m), S(m, l, m)) = 0 \]

and

\[ H(S(n, m, l), S(n, m, l)) = 0.\]

So as \( k \to \infty \) we have

\[ d(k+1, S(l, m, n)) = \in\{d(k+1, a) : a \in S(l, m, n)\}, \]

\[ d(m, S(m, l, m)) = \in\{d(1, b) : b \in S(m, l, m)\}, \]

\[ d(n, S(n, m, l)) = \in\{d(1, c) : c \in S(n, l, m)\}.
\]

Hence there exist sequences \( p_k \in S(l, m, n) \), \( w_k \in S(m, l, m) \) and \( r_k \in S(n, m, l) \) such that \( \lim_{k \to \infty} d(k+1, p_k) = 0 \),

\[ \lim_{k \to \infty} d(m, w_k) = 0 \]

and \( \lim_{k \to \infty} d(n, r_k) = 0.\)

It remains to prove that \( k \to \infty, p_k \to l, w_k \to m, r_k \to n.\)

Suppose that \( p_k \) does not converge to \( l \). Now as \( k \to \infty \)

\[ d(p_k, l) < \lim_{k \to \infty} d(p_k, l) = 0, \]

\[ d(p_k, l) + d(l, l) < d(p_k, l) + d(l, l) .\]

Therefore \( d(p_k, l) < d(p_k, l) \), which is a contradiction. Hence \( \lim_{k \to \infty} p_k = l.\)

Similarly we can prove that \( \lim_{k \to \infty} w_k = m, \lim_{k \to \infty} r_k = n.\)

Since \( S(l, m, n), S(m, l, m) \) and \( S(n, m, l) \) are compact so we have \( l \in S(l, m, n), m \in S(m, l, m) \) and \( n \in S(n, m, l).\)

This shows that \( (l, m, n) \) is a tripled fixed point of \( S.\)

Using Theorem 2.1, we now prove a tripled coincidence and common fixed point theorems for a hybrid pair of multivalued and single valued mapping.

**Theorem 2.2** Let \( (X, d) \) be a complete metric space and let \( S : X \times X \times X \to CB(X) \) be a set valued mapping and \( f : X \to X \) satisfies

\[ H(S(l, m, n), S(a, b, c)) \leq \max \left\{ \zeta \left( \max \{ \zeta(f(La), \zeta(fm, b), \zeta(fn, c) \} \right), \right. \\
\left. \zeta \left( \max \{ \zeta(a, b), \zeta(a, b), \zeta(a, b) \} \right) \right\} \\
\left\{ \max \{ \zeta(l, m), \zeta(m, n) \} \right\}\right].
\]

for some \( a_1 \in S(l, m), a_2 \in S(m, l), a_3 \in S(n, m) \) and \( b_1 \in S(a_1, b), b_2 \in S(b, b), b_3 \in S(c, c) \). Where \( x \in X, X \to [0, \infty) \) is lower semi continuous function and \( \zeta([0, \infty)) \to [0, \infty) \) be an upper semi continuous function.

Further assume that \( S(X \times X \times X) \subseteq f(X) \) Then \( S, f \) have a tripled coincidence point.

Further, \( S, f \) have a tripled common fixed point if one of the following conditions holds.

(a) \( S, f \) is \( w \)-compatible, there exists \( a_1, b_2, c_1 \in X \) such that \( \lim_{k \to \infty} f^k a_1 = a_1, \lim_{k \to \infty} f^{k+1} b_2 = b_2 \) and \( \lim_{k \to \infty} f^{k+1} c_1 = c_1 \) whenever \((l, m, n)\) is tripled coincidence point of \( S, f \) and \( f \) is continuous at \( a_1, b_2, c_1. \)

(b) There exists \( a_1, b_2, c_1 \in X \) such that \( \lim_{k \to \infty} f^k a_1 = a_1, \lim_{k \to \infty} f^{k+1} b_2 = b_2 \) and \( \lim_{k \to \infty} f^{k+1} c_1 = c_1 \) whenever \((l, m, n)\) is a tripled coincidence point of \( (T, f) \) and \( f \) is continuous at \( l, m \) and \( n.\)

**Proof:** By Lemma 1.4, there exists \( E \subseteq X \) such that \( f : E \to X \) is one to one and \( f(E) = f(X) \).

Now define \( T : f(E) \times f(E) \to CB(X) \) by \( T(fL, fn, fM) = f(L, M, N) \) for all \( fL, fn, fM \in f(E). \)

Since \( f \) is one-one on \( E \), so \( T \) is well defined.

Now

\[ H(T(fL, fm, fn), T(fa, fb, fc)) = H(S(l, m, n), S(a, b, c)) \]

\[ \leq \max \left\{ \zeta \left( \max \{ \zeta(fLa, \zeta(fm, b), \zeta(fn, c) \} \right), \right. \\
\left. \zeta \left( \max \{ \zeta(a, b), \zeta(a, b), \zeta(a, b) \} \right) \right\} \\
\left\{ \max \{ \zeta(l, m), \zeta(m, n) \} \right\}\right].
\]

Hence \( T \) satisfies all the conditions and the contraction of Theorem 2.1. So by Theorem 2.1, \( T \) has a tripled fixed point say \((u, v, w) \in f(E) \times f(E) \times f(E). \)

Thus,

\[ a_1 \in T(a_1, b_2, c_1), \]

\[ b_2 \in T(b_2, a_3, b_1), \]

\[ c_1 \in T(c_1, b_1, a_2). \]

Since \( S(X \times X \times X) \subseteq f(X) \), so there exists \( a_2, b_2, c_2 \in X \times X \times X \) such that \( f(a_2) = a_1, f(b_2) = b_2 \) and \( f(c_2) = c_1. \)

Now from the above relation we have

\[ f(a_2) \in T(f(a_2), f(b_2), f(c_2)) = S(a_2, b_2, c_2), \]

\[ f(b_2) \in T(f(b_2), f(a_2), f(b_2)) = S(a_2, b_2, b_2), \]

\[ f(c_2) \in T(f(c_2), f(b_2), f(a_2)) = S(c_2, b_2, a_2). \]

This shows that \( (a_2, b_2, c_2) \in X \times X \times X \) is a tripled coincidence point of \( S, f. \)

Suppose condition (a) holds.
Since \((a_2, b_2, c_2)\) is a tripled coincidence point of \(T\) and \(f\), there exists \(a_2, b_2, c_2 \in X\) such that \(\lim_{k \to \infty} f^{k}a_2 = a_2\), \(\lim_{k \to \infty} f^{k}b_2 = b_2\) and \(\lim_{k \to \infty} f^{k}c_2 = c_2\).

Since \(f\) is continuous at \(a_2, b_2\) and \(c_2\), we have \(f a_2 = a_2, f b_2 = b_2\) and \(f c_1 = c_2\).

Since \(f a_2 \in S(a_2, b_2, c_2)\), we have \(f^2 a_2 \in S(f(a_2, b_2, c_2)) \subseteq S(f(a_2, f(b_2, f(c_2))\).

Since \(f b_2 \in S(b_2, a_2, b_2)\), we have \(f^2 b_2 \in S(f(b_2, f(a_2, b_2)) \subseteq S(f(a_2, f(b_2, f(c_2))\).

Since \(f c_1 \in S(c_2, b_2, a_3)\), we have \(f^2 c_1 \in S(f(c_2, f(b_2, a_3)) \subseteq S(f(a_3, f(b_2, f(c_3))\).

This shows that \((f a_2, f b_2, f c_1)\) is a tripled coincidence point of \(T\) and \(f\).

Similarly, we can prove that \((f^k a_2, f^k b_2, f^k c_2)\) is a tripled coincidence point of \(T\) and \(f\).

Therefore we have
\[
\begin{align*}
\lim_{k \to \infty} & f^k a_2 \in S(f^{k-1} a_2, f^{k-1} b_2, f^{k-1} c_2) \\
\lim_{k \to \infty} & f^k b_2 \in S(f^{k-1} b_2, f^{k-1} a_2, f^{k-1} c_2) \\
\lim_{k \to \infty} & f^k c_2 \in S(f^{k-1} c_2, f^{k-1} b_2, f^{k-1} a_2).
\end{align*}
\]

Now,
\[
\begin{align*}
d(f a_2, S(a_2, b_1, c_1)) & \leq d(f a_2, f^k a_2) + d(f^k a_2, S(a_2, b_1, c_1)) \\
& \leq d(f a_2, f^k a_2) + (f^k a_2, f^k b_2, f^k c_2), S(a_2, b_1, c_1)) \\
& \leq d(f a_2, f^k a_2) \\
& + \max \left\{ \max \{\max(\phi(a_2, b_2, f(a_2, b_2, f(c_2))), \phi(a_2, f(b_2, c_2)), \phi(a_2, b_2, c_2)) \} \\
& - \max \{\max(\phi(a_2, b_2, f(b_2, c_2)), \phi(a_2, b_2, f(a_2, b_2, f(c_2))) \} \\
& \right. \\
& \right\} \\
& \leq d(f a_2, f^k a_2) + \max \left\{ \max \{\max(\phi(a_2, b_2, f(a_2, b_2, f(c_2))), \phi(a_2, b_2, f(a_2, b_2, f(c_2))) \} \\
& - \max \{\phi(a_2, b_2, f(b_2, c_2)), \phi(a_2, b_2, f(a_2, b_2, f(c_2))) \} \right\} \\
& \leq 0.
\end{align*}
\]

Letting \(k \to \infty\), we obtain
\[
\begin{align*}
d(f a_2, S(a_2, b_1, c_1)) & \leq d(f a_2, f a_2) \\
& + \max \left\{ \max \{\phi(a_2, b_2, f(a_2, b_2, f(c_2))), \phi(a_2, b_2, f(a_2, b_2, f(c_2))) \} \\
& - \phi(a_2, b_2, f(b_2, c_2)), \phi(a_2, b_2, f(a_2, b_2, f(c_2))) \right\} \\
& \leq 0.
\end{align*}
\]

which implies that \(f a_2 \in S(a_2, b_1, c_1)\).

Thus \(a_2 = f a_2 \in S(a_2, b_1, c_1)\). In the same way we can prove that \(b_2 = f b_2 \in S(b_1, a_2, c_1)\) and \(c_2 = f c_1 \in S(c_1, b_2, a_2)\).

This shows that \((a_2, b_1, c_1)\) is a tripled common fixed point of the hybrid pair \((S, f)\).

Suppose condition (b) holds.

Since \((a_2, b_2, c_2)\) is a tripled coincidence point of \((S, f)\), there exists \(a_2, b_2, c_2 \in X\) such that \(\lim_{k \to \infty} f^{k}a_1 = a_2, \lim_{k \to \infty} f^{k}b_1 = b_2\) and \(\lim_{k \to \infty} f^{k}c_1 = c_2\).

Since \(f\) is continuous at \(a_2, b_2\) and \(c_2\), we have
\[
\begin{align*}
f a_2 = a_2 & \quad f b_2 = b_2 \quad f c_2 = c_2.
\end{align*}
\]

Thus \(a_2 = f a_2 \in S(a_2, b_2, c_2)\), \(b_2 = f b_2 = S(b_2, a_2, b_2)\) and \(c_2 = f c_2 = S(c_2, b_2, a_2)\).

Hence \((a_2, b_2, c_2)\) is a tripled common fixed point of \((S, f)\).

Hence the results if proved.

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