ISSN-2394-5125 VOL 07, ISSUE 01, 2020

ALGEBRAIC STRUCTURE OF SALINGAROS VEE GROUP OVER CLIFFORD ALGEBRA

Farooqhusain Inamdar[1*], Hasan S.N.[2]

[1*,2]Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, Telangana, India
E-mail:[1]farooqhusain2690@gmail.com , [2]hasan.najam@manuu.edu.in

***Corresponding Author:** Farooqhusain Inamdar
*Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, Telangana, India

Abstract

Rafal Ablamowicz have shown the Classification of Clifford algebra $Cl_{(p,q)}$ as images of group algebra of SalingarosVeegroup $G_{(p,q)}$.Here $G_{(p,q)}$ is a 2-group of order2^{p+q+1} belonging to one of Salingaros isomorphic classes $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}$ and $S_k$which are non-isomorphic to each other and every real Clifford Algebra $Cl_{(p,q)}$ is $\mathbb{R}$ -isomorphic to a quotient of a group algebra $\mathbb{R}[G_{(p,q)}]$. In this paper we show how group structure of Salingaros Vee group $G_{(p,q)}$ in the presence of normal subgroup and central product structure carry over Clifford Algebra $Cl_{(p,q)}$.

Key words: 2-group, central product, Clifford algebra, group algebra, quotient algebra, normal group, Salingaros Vee group

1. Introduction

The Clifford algebras have been developed with the involvement of several Mathematicians and physicists such as *Rudolf Lipschits, Theodor Valen, Elie Cartan, Claude Chevalley* reinvented the "Clifford Algebra" [1] and established its power as a formal mathematics and physics language. Specifically, *David Hesten* and *Elie Cartan* are notable contributors to the progress and development of Clifford algebra. *Elie Cartan* presented the idea of the spinor in 1913 and in 1938 the idea of the pure spinor and he defined Clifford algebra's as algebras of matrices and found that 8 has a periodicity inside these algebraic structures, for more info, refer [2]. *David Hesten* extended the concept of "Clifford Algebra" to devise a formalism and calls it Geometric Algebra [3]. He defines orthogonal operators as similarity transformations on Euclidian space *E*, which can also be considered as group actions in Clifford Algebra on the underlying Vector Space. Salingaros has noted that these groupings are members of five non-isomorphic families [4, 5]. However, one is aware that there are five distinct families into which all Clifford algebras $Cl_{(p,q)}$ may be divided as simple and semi simple algebras depending on the values of *(p, q)* and *p + q* (the Periodicity of Eight)[6, 7, 8]. In this paper we will discuss the algebraic structure of Salingaraos Vee group over the Clifford Algebra$Cl_{(p,q)}$ [9, 10, 11].

2. Preliminaries
2.1 Clifford's original definition

Grassmann's exterior algebra $\wedge \mathrm{R}^n$ of the linear space R^n is an associative algebra of dimension 2^n. In terms of a basis { $\sigma_1, \sigma_2, \sigma_3, \dots\dots, \sigma_n$ } for R^n the exterior algebra $\wedge \mathrm{R}^n$ has a basis,

1

$\sigma_1, \sigma_2, \sigma_3, \dots\dots, \sigma_n$

$\sigma_1 \wedge \sigma_2,\ \sigma_1 \wedge \sigma_3, \dots .\ \sigma_1 \wedge \sigma_n,\ \sigma_2 \wedge \sigma_3, \dots\dots \sigma_(n-1) \wedge \sigma_n$

.

.

.

$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \dots\dots \wedge \sigma_n$

The exterior algebra has unit 1 and satisfies the multiplication rules

$$\sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i \text{ for } i \neq j$$

$$\sigma_i \wedge \sigma_i = 0 \tag{1}$$

Clifford in 1882 kept the first rule but altered the second rule, and arrived at the multiplication rule

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \text{ for } i \neq j$$

$$\sigma_i \sigma_i = 1 \tag{2}$$

This time $\{ \sigma_1, \sigma_2, \sigma_3, \dots \dots, \sigma_n \}$ is an orthonormal basis for the positive definite Euclidean space R^n. An associative algebra of dimension 2^n so defined is the Clifford algebra Cl_n.
Clifford in 1878, considered the multiplication rules

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \text{ for } i \neq j$$
$$\sigma_i \sigma_i = -1 \tag{3}$$

of the Clifford algebra $Cl_{(0,n)}$ of the negative definite space $R^{(0,n)}$ refer [7, 8].

2.2 Clifford Algebra

All real Clifford algebras are defined on the underlying real vector space *n*, the space of vectors with *n* real components on which we are giving a non-degenerated quadratic form $Q = x^t = Qx$. since Q is symmetric, it's eigen values are real and Q is non-degenerated hence it has p positive and q negative eigen values with p + q = n. The pair (p, q) is called the signature of Q and is the only important property of Q in defining the associative Clifford Algebra. This algebra will be written as $Cl_{(p,q)}$ or some times $Cl(Q)$.
The orthonormal basis $\{ \sigma_1, \sigma_2, \sigma_3, \dots \dots, \sigma_n \}$ in E^n relative to Q, so that

$$\sigma_i{}^{\mathrm{T}} Q \sigma_i = 0, \text{for } i \neq j$$

$$\sigma_i{}^{\mathrm{T}} Q \sigma_i = \begin{cases} +1, & i = 1 \dots p \\ -1, & i = p+1 \dots \dots p+q = n \end{cases}, \text{ for } i = j \tag{4}$$

Define multiplication to be an associative operation which satisfies the two conditions,

$$\sigma_i{}^2 = \begin{cases} +1, & i = 1 \dots p \\ -1, & i = p+1 \dots \dots p+q = n \end{cases}, \text{ for} i = j \tag{5}$$

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \text{ for } i \neq j \tag{6}$$

The above equations allow us to define, any product $\sigma_{i_1} \sigma_{i_2} \dots \dots \sigma_{i_s}$for $i_1 < i_2 < \dots . < i_s$, is asubset of {1,2,3,....n}, because from equation (6) we can always reorder the indices, for example,

$$\sigma_1 \sigma_2 \sigma_3 = (\sigma_1 \sigma_2) \sigma_3 = -(\sigma_2 \sigma_1) \sigma_3 = -\sigma_2 (\sigma_1 \sigma_3) = \sigma_2 (\sigma_3 \sigma_1) = (\sigma_2 \sigma_3) \sigma_1 = -\sigma_3 \sigma_2 \sigma_1$$

Therefore the Clifford algebra is a vector space spanned by the product $\sigma_{i_1} \sigma_{i_2} \dots \dots \sigma_{i_s}$, whoseelements can always be written in increasing order [12].
The following formal polynomial represents an arbitrary element $\mathcal{A}$ inthe Clifford Algebra $Cl_{(p,q)}$:

$$\begin{aligned} \mathcal{A} = a^0 \sigma_0 + \textstyle\sum_{i=1}^n a^i \sigma_i + \sum_{i=1}^n \sum_{j=1}^n a^{ij} \sigma_{ij} + \dots + \sum_{i_1=1}^n \dots \sum_{i_k=1}^n a^{i_1 \dots i_k} \sigma_{i_1 \dots i_k} + \\ + \dots + a^{12\dots n} \sigma_{12\dots n} = \textstyle\sum_{k=0}^n a^{i_1 i_2 \dots i_k} \sigma_{i_1 i_2 \dots i_k} \end{aligned} \tag{7}$$

2.3 Fundamental Automorphism of Clifford Algebra

Clifford Algebra $Cl_{(p,q)}$ has four fundamental automorphism, which are as follows, refer [10].

2.3.1 Identity

Let $\mathcal{A}$ be any random element of Clifford Algebra $Cl_{(p,q)}$, the Identity automorphism from $\mathcal{A} \to \mathcal{A}$ is one which carries $\sigma_i \to \sigma_i$.

2.3.2 Involution

Let $\mathcal{A} = \mathcal{A}' + \mathcal{A}''$ be the decomposition of an element of Clifford Algebra $Cl_{(p,q)}$, where $\mathcal{A}'$ and $\mathcal{A}''$ contains homogeneous odd and even components individually, then automorphism $\mathcal{A} \to \mathcal{A}^*$ is the Involution so that the sign of the elements of $\mathcal{A}''$ doesn't change and the sign of elements of $\mathcal{A}'$ changes, i.e,

$$\mathcal{A}^* = -\mathcal{A}' + \mathcal{A}''$$

In general Involution automorphism caries $\sigma_i \to -\sigma_i$, for any element of $\mathcal{A}$.
The Involution automorphism can also be expressed with the help of the volume element ω, i.e $= \sigma_{i_1 i_2} \cdots \cdot_{i_{p+q}}$, such that, $\mathcal{A} = \omega \mathcal{A} \omega^{-1}$, where $\omega^{-1} = (-1)^{\frac{(p+q)(p+q-1)}{2}} \omega$ refer [12, 14].

2.3.3 Reversion

The Reversion of any element of Clifford Algebra $Cl_{(p,q)}$ is the Antiautomorphism from $\mathcal{A} \to \tilde{\mathcal{A}}$, that is an alternative to any basis element $\sigma_{i_1 i_2} \cdots \cdot_{i_k} \in \mathcal{A}$ by an element of$\sigma_{i_k i_{k-1}} \cdots \cdot_{i_1}$, such that:

$$\sigma_{i_1 i_2} \cdots \cdot_{i_k} = (-1)^{\frac{k(k-1)}{2}} \sigma_{i_k i_{k-1}} \cdots \cdot_{i_1},$$

Hence for any element $\mathcal{A}$ of Clifford Algebra $Cl_{(p,q)}$,

$$\tilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}$$

2.3.4 Conjugation

The Conjugation of any element $\mathcal{A}$ of Clifford Algebra $Cl_{(p,q)}$ is the Antiautomorphism from $\mathcal{A} \to \tilde{\mathcal{A}}^*$ which is the composition of Involution and Reversion Automorphism, [11] such that

$$\tilde{\mathcal{A}}^* = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}$$

3. Clifford Algebras as Projections of Group Algebras

3.1 Group Algebra

Let *G* be a finite group with *m* elements, The Group algebra $\mathbb{F}[G]$over the field $\mathbb{F}$ is the linear combinations of finitely many elements of *G* with coefficients in $\mathbb{F}$ i.e,

$$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \cdots + \lambda_m g_m \text{ for } g_i \in G, \lambda_i \in \mathbb{F}, i = 1,2, \ldots m$$

In general, we can rewrite the above linear combination with the algebraic multiplication [6] determined by the group product as follows,

$$\mathbb{F}[G] = \left\{ \sum_{g \in G} \lambda_g g, \lambda_g \in \mathbb{F} \right\}$$

3.2 Group Algebra and their projection on Clifford Algebra

In this paper, we focus only on real group algebras of finite 2-groups, in particular, Salingaros Vee groups.

Definition 1: A group *G* is said to be p-group where *p* is a prime, if every element of group *G* is of order p^k where $k \geq 1$. Therefore any finite group of order p^k is called *p-group.*
As we know quaternion group and the dihedral group are the only group of order eight which are non-abelian groups.

The Quaternion group has the representation $Q_8 = \{a, b | a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\}$. *Vladimir M. Chernov* given another representation of quaternion group Q_8, [13] as follows,

$$Q_8 = \{\varepsilon, \tau, I, J, IJ, \tau I, \tau J, \tau IJ\}$$

which can also be write,

$$= \{I, J, \tau | \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI\}$$

Under the operation of group multiplication, where ε is the identity element of the group $Q_8$the elemen τ is the Involution such that $\tau^2 = \varepsilon$ the element τ is like (-1), with $I = a$, $J = b$, $\tau = a^2$.

The order structure of quaternion group Q_8 is {1, 1, 6} i.e. it has one element of order one, one element of order two and six element of order four. The center ofQ_8 is $Z = (Q_8) = \{1, a^2\}$ which is isomorphic to C_2 [5].

Vladimir M. Chernov's representation of the Dihedral group $D_8 = \{\mathrm{a}, \mathrm{b} | a^4 = b^2 = 1, bab^{-1} = a^{-1}\}$. is as follows,

$$D_8 = \{\varepsilon, \tau, \delta^1, \delta^2, \delta^3, \tau\delta^1, \tau\delta^2, \tau\delta^3\}$$
$$= \{\delta, \tau \mid \delta^4 = \tau^2 = 1, \tau\delta\tau^{-1} = \delta^{-1}\}$$

The group multiplication is defined by $\tau^2 = \varepsilon, \delta^4 = \varepsilon, \tau\delta = \delta^3\tau$, where ε is the identity element of the group D_8 with τ = a, δ = b.

The order structure of Dihedral group D_8 is {1, 5, 2} and center of D_8 is $Z = (D_8) = \{1, a^2\}$. which is isomorphic to C_2.

Now, recall the examples of Clifford Algebra as projection of group algebra from *Vladimir M. Chernov's and Anne Marie Walley* [3, 12, 14] for the construction of $\mathbb{H} \cong Cl_{0,2}$ as $^{E[Q_8]}/_J$ and $Cl_{1,1}$ as $^{E[D_8]}/_J$.

Example 1: Define an $E - algebra\ map\ \psi: E[Q_8] \to \mathbb{H} = \{1, \mathrm{i}, \mathrm{j}, \mathrm{ij}\}$ as follows,

$$1 \to 1, \tau \to -1, I \to i, J \to j$$

Let $u \in E[Q_8]$, then Ker $\psi = \left\{\sum_{g\in Q_8} \lambda_g g, : \psi(u) = 0\right\} = J = (1+\tau)$ for central involution $\tau = a^2$ in Q_8 so that the dim$_E$J = 4 and ψ is bijective [12].

Let $\pi: E[Q_8] \to {}^{E[Q_8]}/_J$ be the natural map $u \to u + J$ the there exists an isomorphism, $\varphi: {}^{E[Q_8]}/_J \to \mathbb{H}$ such that $\varphi o\ \pi = \psi$ by first Isomorphism Theorem (4). It is necessary to verify that $I + J$, $J + J$ fulfill the relations of the generators of $Cl_{0,2}$.

Now Consider the following,

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi\big(\pi(I^2)\big) = \psi(\tau) = -1 = (\psi(I))^2 = \mathbf{i}^2$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi\big(\pi(J^2)\big) = \psi(\tau) = -1 = (\psi(J))^2 = \mathbf{j}^2$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1+\tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi\big(\pi(IJ + JI)\big) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = \mathbf{ij} + \mathbf{ji}$$

Hence, it has been verified that $I + J$, $J + J$ fulfills the relations of the generators of the Clifford Algebra $Cl_{0,2}$. Thus, ${}^{E[Q_8]}/_J \cong \psi(E[Q_8])= \mathbb{H} \cong Cl_{0,2}$, provided the central involution τ is mapped into −1.

Example 2: Define an $E - algebra$ map $\psi: E[D_8] \to Cl_{1,1}$ as follows,

$$1 \to 1, \tau \to \sigma_1 \delta \to \sigma_2,$$

Where $Cl_{1,1}$ spanned by the orthonormal elements $Cl_{1,1} = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ with multiplication relation, ${\sigma_1}^2 = 1, {\sigma_2}^2 = -1, \sigma_1\sigma_2 = -\sigma_2\sigma_1$.

Let $u \in E[D_8]$, then Ker $\psi = \left\{\sum_{g\in D_8} \lambda_g g, : \psi(u) = 0\right\} = J = (1+\delta^2)$ for central involution $\delta^2 = a^2 \in D_8$ so that the dim$_E J$= 4 and ψ is bijective [12].

Let $\pi: E[D_8] \to {}^{E[D_8]}/_J$ be the natural map $u \to u + J$ then there exists an isomorphism, $\varphi: {}^{E[D_8]}/_J \to Cl_{1,1}$ such that $\varphi o\ \pi = \psi$ by first Isomorphism Theorem [4]. It is necessary to verify that $\tau + J$, $\delta + J$ fulfill the relations of the generators of $Cl_{1,1}$.

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi\big(\pi(\tau^2)\big) = \psi(1) = (\psi(\tau))^2 = \sigma_1^2 = 1$$
$$\pi(\delta^2) = \delta^2 + \mathcal{J} \text{ and } \varphi\big(\pi(\delta^2)\big) = \psi(\delta^2) = \psi(-1) = \sigma_2^2 = -1$$
$$\pi(\tau\delta + \delta\tau) = \tau\delta + \delta\tau + \mathcal{J} = \delta\tau(1+\delta^2) + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(\delta\tau + \tau\delta)) = \psi(\tau)\psi(\delta) + \psi(\delta)\psi(\tau) = \psi(0) = \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$$

Hence, it has been verified that $\tau + J$, $\delta + J$fulfills the relations of the generators of the Clifford Algebra $Cl_{1,1}$. Thus, ${}^{E[D_8]}/_J \cong Cl_{1,1}$ provided the central involution δ^2 is mapped into −1.

Example 3: *Anne Marie Walley* extended *Vladimir M. Chernov's* construction to $Cl_{2,0}$ and represented it as $^{E[D_8]}/_J$ as shown in the following example.
Define an $E-algebra$ map: $\psi: E[D_8] \rightarrow Cl_{2,0}$ as follows,

$$1 \rightarrow 1, \tau \rightarrow \sigma_1\delta \rightarrow \sigma_1\sigma_2,$$

Where $Cl_{2,0}$ spanned by the orthonormal elements $Cl_{2,0} = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ with multiplication relation, ${\sigma_1}^2 = 1, {\sigma_2}^2 = 1, \sigma_1\sigma_2 = -\sigma_2\sigma_1$. The further steps are same as in above examples, but here $\tau + J$, $\delta\tau + J$ fulfills the relation of generator of $Cl_{2,0}$ for further details refer [12, 13]. Therefore $^{E[D_8]}/_J \cong Cl_{2,0}$ provided the central involution δ^2 is mapped into -1.

Let us summarize the above three examples for the *Chernov's Theorem* with it's reformulated as follows.
1. The quaternion group Q_8:

$$Q_8 = \{\tau^{\gamma_0} g_1^{\gamma_1} g_2^{\gamma_2} : \gamma_k \in \{0,1\}, k = 0,1,2\}$$

Where $g_1 = a, g_2 = b, \tau = a^2$ is central involution in Q_8. Thus

$${g_1}^2 = a^2 = \tau, {g_2}^2 = b^2 = a^2 = \tau, \tau g_1 g_2 = g_2 g_1$$

Note that the order of elements, $|g_1|$=$|g_2|$=4 and $^{E[Q_8]}/_J \cong Cl_{0,2}$ where $J = (1 + \tau)$

2. The quaternion group D_8:

$$D_8 = \{\tau^{\gamma_0} g_1^{\gamma_1} g_2^{\gamma_2} : \gamma_k \in \{0,1\}, k = 0,1,2\}$$

Where $g_1 = b, g_2 = a, \tau = a^2$ is central involution in D_8. Thus,
${g_1}^2 = b^2 = 1$, ${g_2}^2 = a^2 = \tau, \tau g_1 g_2 = g_2 g_1$

Note that the order of elements, $|g_1|$=2, $|g_2|$=4 and $^{E[D_8]}/_J \cong Cl_{1,1}$ where $J = (1 + \tau)$.

Theorem: Let $Cl_{(p,q)}$ be the universal Clifford Algebra defined as in 1.1 and let G be the finite *2−group* of order 2^{1+n} generated by a central involution τ and additional elements $g_1, g_2, g_3 \ldots. g_n$ which satisfy the following relations [14,15]:

$$\tau^2 = 1, (g_1)^2 = (g_2)^2 = \cdots = (g_p)^2, (g_{p+1})^2 = (g_{p+2})^2 = \cdots = (g_{p+q})^2$$
$$\tau g_i = g_i \tau, g_i g_j = \tau g_j g_i, i, j = 1,2, \ldots n = p + q$$

So that $G = \{\tau^{\gamma_0} g_1^{\gamma_1} g_2^{\gamma_2} \ldots \ldots g_2^{\gamma_n} : \gamma_k \in \{0,1\}, k = 0,1,2 \ldots \ldots n\}$. et J = (1 + τ) be an ideal in the group algebra E[G], then

- $\dim_E J = 2^n$
- Here exists a surjective $E-algebra$ homomorphism $\psi: E[G] \rightarrow Cl_{p,q}$ with the ker $\psi = J$.

4. Salingaros Vee Group

A finite group can be used to describe the Clifford Algebra structure in a vary elegant way. Let $G_{(p,q)}$ be a finite group in any real Clifford Algebra $Cl_{p,q}$,where a binary operation is just the Cifford Algebra product and the basis elements of Clifford Algebra $Cl_{p,q}$ form a finite group of order 2^{1+p+q}. Such that

$$G_{p,q} = \{\pm 1, \pm\sigma_i, \pm\sigma_i\sigma_j, \pm\sigma_i\sigma_j\sigma_k, \ldots \pm \sigma_1\sigma_2\sigma_3 \ldots \sigma_n\} \text{ where } n = p + q$$

$G_{p,q}$ may also represented as follows,

$$G_{p,q} = \langle -1, \sigma_1, \ldots, \sigma_n | \sigma_i\sigma_j = -\sigma_j\sigma_i \text{ for } i \neq j \text{ and } \sigma_i^2 = \pm 1 \rangle$$

Where ${\sigma_i}^2 = 1$ for $1 \leq i \leq p$ and ${\sigma_i}^2 = -1$ for $p + 1 \leq i \leq n = p + q$ and the elements $\sigma_{i\_} = \sigma_{i_1}\sigma_{i_2} \ldots \ldots \sigma_{i_k}$ will be simplified as $\sigma_{i_1 i_2 i_3 \ldots\ldots i_k}$ for $1 \leq k$ while σ_0 will be denoted as 1, the identity element of $G_{p,q}$(and $Cl_{p,q}$).

This 2-group of order 2^{1+p+q} is called Salingarosvee group. Salingaros categorized this group into five isomorphic classes $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}$ and S_k which are non-isomorphic to each other [9, 10, 11].

5. Algebraic Structure of Salingaros Vee Group

5.1 Vee group corresponding to Clifford algebra $Cl_{0,0}$

Now let's look at a few of the groups' most basic instances. Initially, a finite Vee group that is associated with the Clifford algebra $Cl_{0,0}$with an arbitrary element $\mathcal{A} = a^0$ and division ring $\mathbb{K} \approx E(p - q \equiv 0\, mod(8))$ is cyclic group $\mathbb{z} = \{1,1\}$ with the following multiplication table:

	1	-1
1	1	-1
-1	-1	1

It is simple to observe that the finite group that corresponds to Clifford algebra $Cl_{0,0}$ according to Vee group is $N_0 -$ *group* ($N_0 = \mathbb{z}$)

5.2 Vee group corresponding to Clifford algebra $Cl_{1,0}$

Let us take an element of Clifford Algebra$Cl_{1,0}$= $\mathcal{A} = a^0 + a^0\sigma_1$, where ${\sigma_1}^2 = 1$, $\mathbb{K} \approx E \oplus E, (p - q \equiv 1\, mod(8))$. In this case the basis elements of $Cl_{1,0}$ form the *Gauss–Klein four–group* $\mathbb{z} \oplus \mathbb{z} = \{1, -1, \sigma_1, -\sigma_1\}$. The multiplication of the $\mathbb{z} \oplus \mathbb{z}$ is as follows.

	1	-1	σ_1	$-\sigma_1$
1	1	-1	σ_1	$-\sigma_1$
-1	-1	1	$-\sigma_1$	σ_1
σ_1	σ_1	$-\sigma_1$	1	-1
$-\sigma_1$	$-\sigma_1$	σ_1	-1	1

Let $\Omega_k = N_k \otimes \mathbb{Z}_2$. we have here first Ω group:$\Omega_0 = N_0 \otimes \mathbb{Z}_2 = N_0 \otimes N_0 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

5.3 Vee group corresponding to Clifford algebra $Cl_{0,1}$

Let us take an element of Clifford Algebra$Cl_{0,1}$= $\mathcal{A} = a^0 + a^0\sigma_1$, where ${\sigma_1}^2 = -1$, $\mathbb{K} \approx \mathbb{C}, (p - q \equiv 1\, mod(8))$ corresponds to the complex group $\mathbb{z} \oplus \mathbb{z} = \{1, -1, \sigma_1, -\sigma_1\}$ with the multiplication table:

	1	-1	σ_1	$-\sigma_1$
1	1	-1	σ_1	$-\sigma_1$
-1	-1	1	$-\sigma_1$	σ_1
σ_1	σ_1	$-\sigma_1$	-1	1
$-\sigma_1$	$-\sigma_1$	σ_1	1	-1

It is evident that this group is classified by Salingaros as a first $S -$ *group* such that $S_0 = \mathbb{Z}_4$.
Furthermore, we can construct the Vee group corresponding to Clifford Algebra for $Cl_{0,2}$, and Clifford Algebra for $Cl_{2,0}$. From these tables we get the Salingaros Vee group of $N_{odd}, N_{even}, \Omega_{odd}, \Omega_{even}$ and S_k groups.

6. Conclusion

Salingaros has noted that these groupings are members of five non-isomorphic families. However, one is aware that there are five distinct families into which all Clifford algebras $Cl_{p,q}$ may be divided as simple and semisimple algebras depending on the values of *(p, q)* and $p + q$ (the Periodicity of Eight).Through *Chernov's* insight, another relationship with finite Salingaros groups emerges that the algebras $Cl_{p,q}$ can be viewed as images of group algebra, we will look over this later research.

Now we can see that from the multiplication tables. The odd $N -$ *groups* correspond to *real spinors,* for example, N_1 is connected to real *2– spinors,* and N_3 is the group of the real Majorana matrices. The even $N -$ *groups* defines the quaternionic groups. Similarly, the *S –groups* are the 'spinor groups, like S_1 is the group of the complex Pauli matrices, and S_2 is the group of the Dirac matrices. Furthermore, the Ω– *groups* are double copies of the N – *groups* and can be written as a direct product of the $N -$ *groups* with the group of two elements $\mathbb{Z}_2$: $\Omega_k = N_k \otimes \mathbb{Z}_2$.

7. References

1. Hestenes D, Sobczyk G. Clifford algebra to geometric calculus: a unified language for mathematics and physics, Springer Science & Business Media, 2012, vol. 5.
2. Boi L. Clifford geometric algebras, spin manifolds, and group actions in mathematics and physics, Advances in applied Clifford algebras. 2009;19(3):611-656.
3. Delanghe R. Clifford analysis: history and perspective, Computational Methods and Function Theory. 2001;1(1):107-153.
4. Rafal Ablamowicz. On Clifford algebras and related finite groups and group algebras. arXiv preprint arXiv:1610.03583, 2016.
5. Rafal Ablamowicz, Manisha Varahagiri, Anne Marie Walley. A classification of Clifford algebras as images of group algebras of Salingaros Vee groups. Advances in Applied Clifford Algebras. 2018;28(2):1-34.
6. Vaz Jr. J, Da Rocha Jr. R. An introduction to Clifford algebras and spinors. Oxford University Press, 2016.
7. Garling D J. Clifford algebras: an introduction. No. 78, Cambridge University Press, 2011.
8. Hestenes D. New foundations for classical mechanics, Springer Science & Business Media, 2012, Vol. 15.
9. Nikos Salingaros. Realization, extension, and classification of certain physically important groups and algebras. Journal of Mathematical Physics. 1981;22(2):226-232.
10. Nikos Salingaros. On the classification of Clifford algebras and their relation to spinors in n dimensions. Journal of Mathematical Physics. 1982;23(1):1-7.
11. Nikos Salingaros. The relationship between finite groups and Clifford algebras. Journal of mathematical physics. 1984; 25(4):738-742.
12. Lounesto P. Clifford algebras and spinors, Cambridge university press, 2001, Vol. 286.
13. Riesz M. Clifford numbers and spinors, Springer Science & Business Media, 2013, Vol. 54.
14. Shimura G. Arithmetic and analytic theories of quadratic forms and Clifford groups. No. 109, American Mathematical Soc., 2004.
15. Varlamov V. Universal coverings of orthogonal groups, Advances in Applied Clifford Algebras. 2004;14(1):81-168.