Approximating the Parameters of Weibull Density by using Lindley Bayes with Squared Error Loss Function

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Abstract
In life testing problems engineers must often deal with lifetimes data that are non-homogeneous. The two-component Weibull mixture is then a highly relevant model to capture heterogeneity for a large majority of operating lifetimes. Unfortunately, the performance of classical estimation methods is risked due to the high number of parameters. The Weibull mixture parameters estimation, in this research we propose a Bayesian Approximation by Lindley approach to provide the posterior density. In this paper we dealt with the estimation of an unknown scale parameter of the two parameter Weibull distribution with Squared Error loss function suggested in this paper.It deals with the methods to obtain the approximate Bayes estimators of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parameters, squared error Loss function (SELF), are used to obtain the approximate Bayes Estimators. A numerical calculation is done for approximate Bayes estimator and its relative mean squared errors by R programming to present the statistical properties of the estimators.

Keywords: Weibull Density, Lindley Approximation, Bayesian Technique, Squared Error Loss, Censoring.

Introduction

Weibull distribution has been extensively used in life testing and reliability probability problems. The distribution is named after the Swedish scientist Weibull who proposed if for the first time in 1939 in connection with his studies on strength of material. Weibull (1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many authors [see Kao (1959)].

The probability density function reliability and hazard rate functions of Weibull distribution are given respectively as

$$f(x) = \mu\nu x^{(\mu-1)} \exp(-\mu\, x^{\mu}) \quad ; \quad x, \nu, \mu > 0 \tag{1.1}$$

$$R(t) = exp(-\nu t^{\mu}) \quad ; \quad t > 0 \tag{1.2}$$

$$H(t) = \mu\nu t^{(\mu-1)} \quad ; \quad t > 0 \tag{1.3}$$

Where $'\nu'$ is the scale and 'μ' is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta},\theta) = k(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$(1.4)

known as squared error loss function (SQELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson(1985). Canfield (1970), Basu and Ebrabimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressedby prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data into the prior distribution using the Bayes theorem, the first theorem of inference. Hence, we update the prior distribution in the light of observed data. Thus, the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution.

The Estimators

Let $x_1, x_2, \dots\dots\dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1.1). The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= (x_{(1)}, x_{(2)}, \dots\dots\dots\dots\dots, x_{(n)})$, where $x_{(i)}$ is the life time of the ith item. Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$L(x|\nu,\mu) = \frac{n!}{(n-r)!}(\mu\nu)^r \prod_{i=1}^{r} x_i^{(\mu-1)} \exp(-\delta\nu)$,(2.1)

Where$\delta = \sum_{i=1}^{r} x_i^{\mu} + (n-r)x_r^{\mu}$

The logarithm of the likelihood function is

$\log L(x|\nu,\mu) \propto r \log\mu + r\log\nu + (\mu-1)\sum_{i=1}^{r}\log x_i - \delta\nu$,(2.2)

assuming that 'μ' is known, the maximum likelihood estimator $\hat{\nu}_{ML}$of ν can be obtain by using equation (2.2) as

$\hat{\nu}_{ML} = r/\delta$(2.3)

In case if both the parameters μ and ν are unknown their MLE's $\hat{\mu}_{ML}$and $\hat{\nu}_{ML}$ can be obtained by solving the following equation

$\frac{\delta}{\delta\nu}\log L = \frac{r}{\nu} - \delta = 0$, (2.4a)

$\frac{\delta \log L}{\delta\mu} = \frac{r}{\mu} + \sum_{i=1}^{r}\log x_i - \nu\delta_1 = 0$,(2.4b)

where

$\delta_1 = \sum_{i=1}^{r} x_i^{\mu} \log x_i + (n-r) x_r^{\mu} \log x_r$, eliminating ν between the two equations of (2.4) and simplifying we get

$\hat{\mu}_{ML} = \frac{r}{\delta^*}$(2.5)

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} log x_i\right]$

Equation (2.5) may be solved for Newton-Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with μ get $\hat{\mu}$ as

$$\hat{\nu}_{ML} = \frac{\frac{r}{\hat{\mu}_{ML}} + \sum_{i=1}^{r} log x_i}{\sum_{i=1}^{r} x_i^{\hat{\mu}_{ML}} \; log x_i + (\text{n}-\text{r}) x_r^{\hat{\mu}_{ML}} \; log x_r}, (2.6)$$

Bayes Estimator of Scale Parameter ν when shape Parameter μ is known :

If μ is known assume gamma prior $\rho(c, d)$ as cojugate prior for ν as

$$\phi(\nu|\underline{x}) = \frac{d^c}{\Gamma c} (\text{v})^{(c+1)} \exp(-\text{dv}) \, ; (c, d) > 0, \nu > 0 \;\; ,(3.1)$$

The posterior distribution of ν using equation (2.1) and (3.1) we get

$$\psi(\theta|\underline{x}) = \frac{(\delta+d)^{r+c}}{\Gamma(r+c)} (\text{v})^{(r+c-1)} \exp\big(-\text{v}(\delta + d)\big) , (3.2)$$

Under squared error loss function, the Bayes estimator $\hat{\nu}_{BSQ}$, is the posterior mean given by

$$\hat{\nu}_{BSQ} = \frac{(r+c)}{(\delta+d)} (3.3)$$

4.The Bayes estimators with ν and μ unknown:

The joint prior density of ν and μ is given by

$$\phi^*(\nu|p) = \phi_1(\nu|\mu).\phi_2(\mu)$$

$$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi} p^{-\xi} \text{v}^{(\xi-1)}. \exp\left[-\left(\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right)\right] \;\; ; (\nu, \mu, \lambda, \xi) > 0, (4.1)$$

where

$$\phi_1(\nu|\mu) = \frac{1}{\Gamma\xi} \nu^{-\xi} \text{v}^{(\xi-1)}. \exp\left[-\frac{\nu}{\mu}\right] ; (4.2)$$

And

$$\phi_2(\mu) = \frac{1}{\lambda} \exp\left(-\frac{\mu}{\lambda}\right) \quad ; \; (4.3)$$

The joint posterior density of ν and μ is

$$\psi^*(\nu, \mu|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi} p^{-\lambda} \text{v}^{(\xi+1)} \exp\left[-\left\{\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right\}\right] (\text{v}\mu)^r \prod_{i=1}^{r} x_i^{(\mu-1)} e^{-\mu\nu}}{\iint \frac{1}{\lambda\Gamma\xi} \mu^{(r-\xi)} \text{v}^{(r+\xi+1)} \prod_{i=1}^{r} x_i^{(\mu-1)} . \exp\left[-\left\{\frac{\nu}{\mu} + \frac{\mu}{\lambda} + \mu\text{v}\right\}\right] d\nu d\mu} ; \quad (4.4)$$

Approximate Bayes Estimators

The Bayes estimators of a function $\rho = \rho(\nu, \mu)$ of the unknown parameter $\nu \; and \; \mu$ under squared error loss is the posterior mean

$$\hat{\varrho}_{ABS} = E(\mu|\underline{x}) = \frac{\iint \phi(\nu\mu)\phi^*(\nu,\mu|\underline{x}) d\nu d\mu}{\iint \phi^*(\nu,\mu|\underline{x}) . d\nu d\mu} \; ; (4.5)$$

By using Lindley approximationmethod we evaluate equation(4.5)

$$E(\varrho(\nu, \mu)|\underline{x}) = \frac{\int \phi(\nu) . e^{(l(\nu)+\varrho(\nu))} d\nu}{\int e^{(l(\nu)+\varrho(\nu))} d\nu} ; (4.6)$$

Where $l(\nu) = \log \phi(\nu)$, and $\phi(\nu)$ is an arbitrary function of ν and $l(\nu)$ is the logarithm likelihood function

The Lindley approximation(Lindley (1980)) for two parameter is

$E\left(\hat{\varrho}(\nu,\mu)\middle|\underline{x}\right) = \varrho(\nu,\mu) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}]$, (4.7)

Where

$$A = \sum_{1}^{2}\sum_{1}^{2} \varrho_{ij}\sigma_{ij} \; ; \quad (4.7a)\, , l_{\eta\epsilon} = \left(\delta^{(\eta+\epsilon)} l \middle| \delta\nu_1^{\eta}\delta\nu_2^{\epsilon}\right); \; (4.7b)$$

where$(\eta + \epsilon) = 3$ for $i, j = 12$ $\rho_i = (\delta\rho|\delta\theta_i);$ $(4.7c)$

$$\varrho_i = \frac{\delta\varrho}{\delta\nu_i} \; ; (4.7d) \; , \quad \varrho_{ij} = \frac{\delta^2\varrho}{\delta\nu_i\delta\nu_j} \; ; \forall i \neq j \; ; \; (4.7e)$$

$$A_{ij} = \varrho_i\sigma_{ij} + \varrho_j\sigma_{ji} \; ; \quad (4.7f), \; B_{ij} = \left(\varrho_i\sigma_{ii} + \varrho_j\sigma_{ij}\right)\sigma_{ii} \; ; \; (4.7g),$$

$$C_{ij} = 3\varrho_i\sigma_{ii}\sigma_{ij} + \varrho_j\left(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2\right); (4.7\, j)$$

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i, j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\nu_i\delta\nu_j}$.

All the function in (4.7a - 4.7j) are evaluated at MLE of (ν_1, ν_2).In our case $(\nu_1\nu_2) = (\nu,\mu); So\ \phi(\nu) = \phi(\nu,\mu)$

To apply Lindley approximation (4.5), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i, j = 1,2$,which can be shown to be

$\sigma_{11} = \frac{M}{D},\ \sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}, \sigma_{22} = \frac{r}{D\,\theta^2}$,(4.8a)

Where$M = \left(\frac{r}{\mu^2} + \nu\delta_2\right); D = \left[\frac{r}{\nu^2}\left(\frac{r}{\mu^2} + \nu^2\delta_2\right)\right];$ (4.8b)

$\delta_2 = \sum_{i=1}^{r} x_i^{\mu}\,(log x_i)^2 + (n-r)x_r^{\mu}(log x_r)^2;$ (4.8c)

To evaluateρ_i , take the joint prior $\phi^*(\nu|\mu)$

$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi}\mu^{-\xi}\nu^{(\xi-1)}.\exp\left[\left\{-\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right\}\right] \; ; (\nu,\mu,\lambda,\xi) > 0$, (4.9)

$$\Rightarrow \rho = \log[\phi^*(\nu|\mu)] = constant - \xi log\mu - (\xi - 1)log\nu - \frac{\nu}{\mu} - \frac{\mu}{\lambda}$$

Therefore

$\rho_1 = \frac{\partial\rho}{\partial\nu} = \frac{(\xi-1)\nu}{\nu} - \frac{1}{\nu}$;(4.9a)

and

$\rho_2 = \frac{\nu}{\mu^2} - \frac{1}{\lambda} - \frac{\xi}{\mu}$;(4.9b)

Further more

$l_{21} = 0\, ; l_{12} = -\delta_2\, ; l_{03} = \frac{2r}{p^3} - \nu\delta_3;$ (4.9c)

$and\ \ l_{30} = \frac{2r}{\nu^3};$ (4.9d)

Where $\delta_3 = \sum_{i=1}^{r} x_i^{\nu}\,(log x_i)^3 + (n-r)x_r^{\nu}(log x_r)^3$

By substituting above values in eqn. (4.7), yields the Bayes estimator under SELF using Lindley approximation denoted by $\hat{\varrho}_{ABS}$

$$\hat{\varrho}_{ABSQ} = E\left(\varrho(\nu,\mu)\right) = \varrho(\nu,\mu) + \mathrm{U} + \varrho_1 U_1 + \varrho_2 U_2; \qquad (4.10)$$

Where $\mathrm{U} = \frac{1}{2}[\varrho_{11}\sigma_{11} + \varrho_{21}\sigma_{21} + \varrho_{12}\sigma_{12} + \varrho_{22}\sigma_{22}]$;(4.10a)

$$\mathrm{U}_1 = \frac{1}{\nu^2 D^2}\Big[\frac{M\nu D}{\mu}(\mu(\xi-1)-1) + \frac{\nu^2\delta_1 D}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\} + \frac{rM^2}{\nu} - \frac{rM\delta_1}{2} - \nu^2\delta_1^2\delta_2 + \frac{r^2}{\nu^3}\delta_1 - \frac{\nu r\delta_1\,\delta_3}{2}\Big]; \qquad (4.10b)$$

$$\mathrm{U}_2 = \frac{1}{\nu^2 D^2}\Big[\frac{\nu\ \ \delta_1 D}{\mu}(\mu(\xi-1)-\nu) + \frac{\mathrm{r}D}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\} + \frac{rM\delta_1}{\nu} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\nu^2\mu^3} - \frac{r^2\delta_3}{2\nu}\Big]; \qquad (4.10c)$$

All the function of right hand side of the equation (4.10) are to be evaluated for $\hat{\nu}_{ML}$ and $\hat{\mu}_{ML}$.

Approximate Bayes Estimates Under Squared Error Loss function

with equations(4.10)-(4.10c), the different Approximate Bayes estimators Under SQELF using Lindley's approximation given by

Special cases:

(substituting $\varrho(\nu,\mu) = \nu$ in equation(4.7), we get the Approximate Bayes Estimator of ν as

$\hat{\nu}_{ABSQ} = \nu + U_1$; at$(\hat{\nu}_{ML}, \hat{\mu}_{ML})$(4.11)

Numerical Calculations and Comparison

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1. The values of $(\nu,\mu)$and are generated from the equations (4.3-4.4) for given c=2, and d=3, which comes out to be ν=0.238 and μ=0.227. For these values of $\nu$and μ the Weibull random variates are generated.

2. Taking the different sizes of samples n=25 (25) 100 with failure censoring, MLE's, the Approximate Bayes estimators,and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the table(1) forparameters of prior distribution c =2 , and d=3.

Table(1)

Mean and MSE's of ν

$(\lambda = 2, \xi = 3, = .238, \mu = .227, a = 20)$

n	r	$\hat{\nu}_{ML}$	$\hat{\nu}_{BSQ}$	$\hat{\nu}_{ABSQ}$
25	20	0.02417	0.028051	0.027771
		(8.2278x10^{-5})	**(7.9161x10^{-5})**	**(7.9384x10^{-5})**
50	30	0.070689	0.078036	0.075839
		(4.8866x10^{-5})	**(4.4380x10^{-5})**	**(4.5669x10^{-5})**
75	50	0.835593	0.822443	0.773257
		(7.4077x10^{-4})	**(7.09160x10^{-4})**	**(5.9679x10^{-4})**
100	75	0.801292	0.796716	0.755629
		(6.5996x10^{-4})	**(6.4915x10^{-4})**	**(5.5588x10^{-4})**

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