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Review Article

SOME APPLICATIONS VIA COMMON COUPLED fiXED POINT THEOREMS IN BIPOLAR METRIC SPACES

G. N. V. Kishore[1], D. Ram Prasad[2], B. Srinuvasa Rao[3], and V. S. Baghavan[4]

[1]**Department of Engineering Mathematics, SRKR Engineering College, Bhimavaram 534204, Andhra Pradesh, India.**
[2]**Department of Department of Mathematics, KL University, Vaddeswaram, Guntur-522502, Andhra Pradesh, India.**
[3]**Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam - 532410, Andhra Pradesh, India.**
[4]**Department of Department of Mathematics, KL University, Vaddeswaram, Guntur-522502, Andhra Pradesh, India.**

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Abstract
In this paper, we establish the existence of common coupled fixed point results for two covariant mappings in Bipolar metric spaces. Some interesting consequences of our results are achieved. Also, we give an illustration which presents the applicability of the achieved results. Moreover, we give an application of nonlinear integral equations as well as Homotopy theory by using fixed point theorems.

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INTRODUCTION AND PRELIMINARIES

Fixed point theory is a beautiful mixture of Analysis, Topology and Geometry. Fixed points Theory has been playing a vital role in the study of nonlinear phenomena. In 1922, Banach[1] proved a fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banach fixed point theorem or contraction mapping principle. In 1987, Guo and Lakshmikantham [2] initiated the notion of coupled fixed point and they established some coupled fixed point results. Afterward, The coupled fixed point theorems in partially ordered metric spaces has been developed by Bhaskar and Lakshmikantham [3]. Subsequently, many authors established coupled fixed point theorems in different spaces (see [4]-[13]).

This work is motivated by the recent work on extension of Banach contraction principle on Bipolar metric spaces, which has been done by Mutlu and Gu¨rdal [14]. Also, they investigated some fixed point and coupled fixed point results on this spaces (see [14],[15]).

The aim of this paper is to initiate the study of a common coupled fixed point results for two covariant mappings under various contractive conditions in bipolar metric spaces. We have also illustrated the validity of the hypotheses of our results.

Definition 1.1 ([14]) Let A and B be a two nonempty sets. Suppose that d : A × B → [0, ∞)

is a mapping satisfying the following properties:

(B_0) If $d(a,b) = 0\ then a = b forall (a,b) \epsilon A \times B$,

(B_1) If $a = b then d(a,b) = 0\ forall (a,b) \epsilon A \times B$,

(B_2) If $d(a,b) = d(b,a) foralla, b \epsilon A \cap B$.

(B_3) If $d(a_1,b_2) \le d(a_1,b_1) + d(a_2,b_1) + d(a_2,b_2) foralla_1, a_2 \epsilon A,\ b_1, b_2 \epsilon B$.

Then the mapping d is called a bipolar-metric on the pair (A, B) and the triple (A, B, d) is called a bipolar-metric space.

Definition 1.2 ([14]) Assume (A1, B1) and (A2, B2) as two pairs of sets.

The function F : A1 ∪ B1 → A2 ∪ B2 is said to be a covariant map, if F (A1) ⊆ A2 and F (B1) ⊆ B2 and denote this as F : (A1, B1) ⇉ (A2, B2).

The mapping F : A1 ∪ B1 ⟶ A2 ∪ B2 is said to be a contravariant map, if F (A1) ⊆ B2 and F (B1) ⊆ A2 and this as F: (A1, B1) ⇋ (A2, B2).

In particular, if d1 and d2 are bipolar metrics in (A1, B1) and (A2, B2) respectively. Then in some times we use the notations F : (A1, B1, d1) ⇉ (A2, B2, d2) and F : (A1, B1, d1) ⇋ (A2, B2, d2).

Definition 1.3. ([11]) Assume (A, B, d) be a bipolar metric space. A point v∈ A ∪ B is termed as a left point if v∈ A, a right point if v∈ B and a central point if both.

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence.

A sequence $\{v_n\}$ as considered convergent to a point$v$, if and only if $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n, v) = 0; or\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\},\{b_n\})$ on (A, B, d) is sequence on the set A × B. If the sequence $\{\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\},\{b_n\})$is said to be convergent. $(\{a_n\},\{b_n\})$ is Cauchy sequence,

if $\lim_{n,m\to\infty} d(a_n, b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 1.4 ([15]) Let (A, B, d) be a bipolar metric space, F:(A2, B2) ⇉ (A, B) be a covariant mapping. If F (a, b) = a and

$F(b, a) = b$ for $(a, b) \in A^2 \cup B^2$ then (a, b) is called a coupled fixed point of F.

MAIN RESULTS

In this section, we give some common fixed point theorems for two covariant mappings satisfying various contractive conditions in bipolar metric spaces.

Definition 2.1 Let (A, B, d) be a bipolar metric space, $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f : (A, B) \rightrightarrows (A, B)$ be two covariant mappings. An element (a, b) is said to be a coupled coincident point of F and f. If $F(a, b) = fa$ and $F(b, a) = fb$.

Definition 2.2 Let (A, B, d) be a bipolar metric space, $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f : (A, B) \rightrightarrows (A, B)$ be two covariant mappings. An element (a, b) is said to be a common coupled fixed point of F and f. If $F(a, b) = fa = a$ and $F(b, a) = fb = b$.

Definition 2.3 Let (A, B, d) be a bipolar metric space, $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f : (A, B) \rightrightarrows (A, B)$ be two covariant mappings are called ω-compatible if $f(F(a, b)) = F(fa, fb)$ and $f(F(b, a)) = F(fb, fa)$ whenever $F(a, b) = fa$ and $F(b, a) = fb$.

Theorem 2.4 Let (A, B, d) be a bipolar metric spaces, suppose that $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f : (A, B) \rightrightarrows (A, B)$ be a covariant mappings satisfying:

(2.4.1) $d(F(a, b), F(p, q)) \leq \theta \max\{d(fa, fp), d(fb, fq),\}$

for all $a, b \in A$ and $p, q \in B$ with $\theta \in (0, 1)$

(2.4.2) $F(A^2 \cup B^2) \subseteq f(A \cup B)$.

(2.4.3) Either (F, f) is ω-compatible.

(2.4.4) $f(A \cup B)$ is complete.

Then the mappings $F : A^2 \cup B^2 \to A \cup B$ and $f : A \cup B \to A \cup B$ have a unique common fixed point of the form (u, u).

Proof. Let $a_0, b_0 \in A$ and $p_0, q_0 \in B$ and from (2.4.2), we construct the bisequence $(\{a_{2n}\}, \{p_{2n}\})$, $(\{b_{2n}\}, \{q_{2n}\})$, $(\{\omega_{2n}\}, \{\chi_{2n}\})$ and $(\{\xi_{2n}\}, \{\kappa_{2n}\})$ in (A, B) as

$$F(a_{2n}, b_{2n}) = fa_{2n+1} = \omega_{2n}$$
$$F(p_{2n}, q_{2n}) = fp_{2n+1} = \chi_{2n}$$
$$F(b_{2n}, a_{2n}) = fb_{2n+1} = \xi_{2n}$$
$$F(q_{2n}, p_{2n}) = fq_{2n+1} = \kappa_{2n}$$

for $n = 0, 1, 2, \ldots$.

Now from (2.4.1), we have

$$d(\omega_{2n}, \chi_{2n+1}) = d\big(F(a_{2n}, b_{2n}), F(p_{2n+1}, q_{2n+1})\big)$$
$$\leq \theta \max\{d(fa_{2n}, fp_{2n+1}), d(fb_{2n}, fq_{2n+1})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n}), d(\xi_{2n-1}, \kappa_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n}), d(\xi_{2n-1}, \kappa_{2n})\} \quad (2.4.5)$$

and

$$d(\xi_{2n}, \kappa_{2n+1}) = d\big(F(b_{2n}, a_{2n}), F(q_{2n+1}, p_{2n+1})\big)$$
$$\leq \theta \max\{d(fb_{2n}, fq_{2n+1}), d(fa_{2n}, fp_{2n+1})\}$$
$$\leq \theta \max\{d(\xi_{2n-1}, \kappa_{2n}), d(\omega_{2n-1}, \chi_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n}), d(\xi_{2n-1}, \kappa_{2n})\} \quad (2.4.6)$$

Combining (2.4.5) and (2.4.6), we get that

$$\max\{d(\omega_{2n}, \chi_{2n+1}), d(\xi_{2n}, \kappa_{2n+1})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n}), d(\xi_{2n-1}, \kappa_{2n})\}$$
$$\leq \theta^2 \max\{d(\omega_{2n-2}, \chi_{2n-1}), d(\xi_{2n-2}, \kappa_{2n-1})\}$$

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$$\leq \theta^{2n} \max\{d(\omega_0, \chi_1), d(\xi_0, \kappa_1)\}.$$

Thus $d(\omega_{2n}, \chi_{2n+1})$

$$\leq \theta^{2n} \max\{d(\omega_0, \chi_1), d(\xi_0, \kappa_1)\} d(\xi_{2n}, \kappa_{2n+1})$$
$$\leq \theta^{2n} \max\{d(\omega_0, \chi_1), d(\xi_0, \kappa_1)\} \quad (2.4.7)$$

On the other hand

$$d(\omega_{2n+1}, \chi_{2n})$$
$$= d(F(a_{2n+1}, b_{2n+1}), F(p_{2n}, q_{2n}))$$
$$\leq \theta \max\{d(fa_{2n+1}, fp_{2n}), d(fb_{2n+1}, fq_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n}, \chi_{2n-1}), d(\xi_{2n}, \kappa_{2n-1})\} \quad (2.4.8)$$

And

$$d(\xi_{2n+1}, \kappa_{2n})$$
$$= d(F(b_{2n+1}, a_{2n+1}), F(q_{2n}, p_{2n}))$$
$$\leq \theta \max\{d(fb_{2n+1}, fq_{2n}), d(fa_{2n+1}, fp_{2n})\}$$
$$\leq \theta \max\{d(\xi_{2n}, \kappa_{2n-1}), d(\omega_{2n}, \chi_{2n-1})\} \quad (2.4.9)$$

Combining (2.4.8) and (2.4.9), we get that

$$\max\{d(\omega_{2n+1}, \chi_{2n}), d(\xi_{2n+1}, \kappa_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n}, \chi_{2n-1}), d(\xi_{2n}, \kappa_{2n-1})\}$$
$$\leq \theta^2 \max\{d(\omega_{2n-1}, \chi_{2n-2}), d(\xi_{2n-1}, \kappa_{2n-2})\}$$

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$$\leq \theta^{2n} \max\{d(\omega_1, \chi_0), d(\xi_1, \kappa_0)\}.$$

Thus

$$d(\omega_{2n+1}, \chi_{2n})$$
$$\leq \theta^{2n} \max\{d(\omega_1, \chi_0), d(\xi_1, \kappa_0)\} d(\xi_{2n+1}, \kappa_{2n})$$
$$\leq \theta^{2n} \max\{d(\omega_1, \chi_0), d(\xi_1, \kappa_0)\} \quad (2.4.10)$$

Moreover,

$$d(\omega_{2n}, \chi_{2n})$$
$$= d(F(a_{2n}, b_{2n}), F(p_{2n}, q_{2n}))$$
$$\leq \theta \max\{d(fa_{2n}, fp_{2n}), d(fb_{2n}, fq_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n-1}), d(\xi_{2n-1}, \kappa_{2n-1})\} \quad (2.4.11)$$

and

$$d(\xi_{2n}, \kappa_{2n})$$
$$= d\big(F(b_{2n}, a_{2n}), F(q_{2n}, p_{2n})\big)$$
$$\leq \theta \max\{d(fb_{2n}, fq_{2n}), d(fa_{2n}, fp_{2n})\}$$
$$\leq \theta \max\{d(\xi_{2n-1}, \kappa_{2n-1}), d(\omega_{2n-1}, \chi_{2n-1})\} \quad (2.4.12)$$

Combining (2.4.11) and (2.4.12), we get

$$\max\{d(\omega_{2n}, \chi_{2n}), d(\xi_{2n}, \kappa_{2n})\}$$
$$\leq \theta \max\{d(\omega_{2n-1}, \chi_{2n-1}), d(\xi_{2n-1}, \kappa_{2n-1})\}$$
$$\leq \theta^2 \max\{d(\omega_{2n-2}, \chi_{2n-2}), d(\xi_{2n-2}, \kappa_{2n-2})\}$$

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$$\leq \theta^{2n} \max\{d(\omega_0, \chi_0), d(\xi_0, \kappa_0)\}.$$

Thus, $d(\omega_{2n}, \chi_{2n}) \leq \theta^{2n} \max\{d(\omega_0, \chi_0), d(\xi_0, \kappa_0)\}$

d (ξ2n, κ2n) ≤ θ2n max {d (ω0, χ0) , d (ξ0, κ0)}

Using the property (B4),

d (ω2n, χ2m) ≤ d (ω2n, χ2n+1) + d (ω2n+1, χ2n+1)

+ · · · + d (ω2m−1, χ2m)

d (ξ2n, κ2m)≤ d (ξ2n, κ2n+1) + d (ξ2n+1, κ2n+1)

+.....+d(ξ2m−1,κ2m)

and

d (ω2m, χ2n) ≤ d (ω2m, χ2m−1) + d (ω2m−1, χ2m−1)

+ · · ·+d(ω2n+1, χ2n)

d (ξ2m, κ2n) ≤ d (ξ2m, κ2m−1) + d (ξ2m−1, κ2m−1)
+ · · · + d (ξ2n+1, κ2n) ,

for each n, m ∈ N with n < m. Then from (2.4.7), (2.4.10), (2.4.13), (2.4.14) and (2.4.15), we have

d (ω2n, χ2m) + d (ξ2n, κ2m)

≤ (d (ω2n, χ2n+1) + d (ξ2n, κ2n+1)) + (d (ω2n+1, χ2n+1) + d (ξ2n+1, κ2n+1))

+ · · · + (d (ω2m−1, χ2m−1) + d (ξ2m−1, κ2m−1))

+(d (ω2m−1, χ2m) + d (ξ2m−1, κ2m))

≤ 2(θ2n + θ2n+1 + · · · + θ2m−1) max { d({ω0, χ1), d(ξ0, κ1) }+ 2(θ2n+1 + θ2n+2 + · · · + θ2m−1) max { d({ω0, χ1), d(ξ0, κ1) }

$\leq 2\,\frac{\theta^{2n}}{\theta-1}\max\{d(\{\omega0,\chi1),d(\xi0,\kappa1)\ \}$

$+\ 2\,\frac{\theta^{2n-1}}{1-\theta}\max\{d(\{\omega0,\chi_0),d(\xi0,\kappa0)\ \}\ \rightarrow 0\ asn\ \rightarrow\infty$

Similarly, we can prove (d (ω2m, χ2n) + d (ξ2m, κ2n)) → 0 as n, m →∞. This shows (ω2n, χ2m) and (ξ2n, κ2m) are Cauchy bisequences in (A, B). Therefore,

$$\lim_{n\to\infty}(\omega_{2n},\chi_{2m}) = \lim_{n\to\infty}((\xi 2n,\kappa 2m)) = 0$$

Since f (A ∪ B) is a complete subspace of (A, B, d), so {ω2n+1}, {χ2m+1}, {ξ2n+1}, {κ2m+1} ⊆ f (A ∪ B) are converges in the complete bipolar metric space (f (A), f (B), d). Therefore, there exist u, v ∈ f (A) and w, z ∈ f (B) with

$\lim_{n\to\infty}\omega_{2n+1} = w,\ \lim_{n\to\infty}\xi 2n+1 = z,$

$\lim_{n\to\infty}\chi_{2n+1} = u,\ \lim_{n\to\infty}\kappa 2n+1 = v$ (2.4.16)

Since f : A ∪ B → A ∪ B and u, v ∈ f (A), w, z ∈ f (B), there exist l, m ∈ A, r, s ∈ B such that fl = u, fm = v and fr = w, fs = z.

From (2.4.1) and (B4), we have
d (F (l, m), w)

≤ d (F (l, m), χ2n+1) + d (ω2n+1, χ2n+1) + d (ω2n+1, w)

≤ d (F (l,{m), F (p2n+1, q2n+1))+d (ω2n+1}, χ2n+1)

+d (ω2n+1, w)

$\leq\theta$ max$\{d(fl,fp_{2n+1}),d(fm,fq_{2n+1})\}$+d (ω2n+1, χ2n+1)

+ d (ω2n+1, w)

≤ θ max {d(fl, χ2n), d(fm, κ2n)}+d (ω2n+1, χ2n+1)
+ d (ω2n+1, w) → 0 as n → ∞

Therefore, d (F (l, m), w) = 0 implies F (l, m) = w = fr.

Similarly, we can prove that F (m, l) = z = fs, F (r, s) = u = fl and

F (s, r) = v = fm. Since (F, f) are ω-compatible mappings, we have F (u, v) = fu, F (v, u) = fv

and F (w, z) = fw, F (z, w) = fz. (2.4.13)

We prove that fu = u, fv = v and fw = w, fz = z.

Now

$d\ (fu,\chi 2n)\ \ =\ d\ (F\ (u,v),F\ (p2n,q2n$
$\leq\ \theta$ max $\{d(fu,fp2n),d(fv,fq2n)\}$

$\leq\ \theta$ max $\{d(fu,\chi 2n-1),d(fv,\kappa 2n-1)\}$

letting n → ∞, we get (2.4.14)

$d\ (fu,u)\ \leq\ \theta$ max $\{d(fu,u),d(fv,v)\}$ (2.4.17)

and $d\ (fv,\kappa 2n)\ \ =\ d\ (F\ (v,u),F\ (q2n,p2n))$

≤ θ max {d(fv, fq2n), d(fu, fp2n)}

$\leq\ \theta$ max $\{d(fv,\kappa 2n-1),d(fu,\chi 2n-1)\}$

(2.4.15)

letting n → ∞, we get

d(fv, v) ≤ θ max {d(fv, v), d(fu, u)}

combining (2.4.17) and (2.4.18), we have

max {d(fu, u), d(fv, v)}≤ θ max {d(fu, u), d(fv, v)}

< max {d(fu, u), d(fv, v)}

this holds only possible,d(fu, u) = 0, d(fv, v) = 0 implies fu = u, fv = v. Similarly, we can show fw = w and fz = z. Therefore,

F (w, z) = fw = w = fr = F (l, m)

F (z, w) = fz = z = fs = F (m, l)

F (u, v) = fu = u = fl = F (r, s)

F (v, u) = fv = v = fm = F (s, r).

On the other hand, from (2.4.16), we get

$$d(fl,fr) = d(u,w) = d\left(\lim_{n\to\infty}\chi_{2n},\lim_{n\to\infty}\omega_{2n}\right)$$

$$= \lim_{n\to\infty}(\omega_{2n},\chi_{2n}) = 0$$

And

$$d(fm,fs) = d(v,z) = d\left(\lim_{n\to\infty}\kappa_{2n},\lim_{n\to\infty}\xi_{2n}\right)$$

$$= \lim_{n\to\infty}(\xi_{2n},\kappa_{2n}) = 0$$

So u = w and v = z. Therefore, (u, v) ∈ A2 ∩B2 is coupled fixed point of covariant mappings F and f .

Now we prove the uniqueness, we begin by taking (u∗, v∗) ∈ A2 ∪ B2 be another fixed point of F and f .

If (u∗, v∗) ∈ A2, then we have

d (u, u∗) = d(F (u, v), F (u∗, v∗))

≤ θ max {d(fu, fu∗), d(fv, fv∗)}

≤ θ max {d(u, u∗), d(v, v∗)} (2.4.19)

and d (v, v∗) = d(F (v, u), F (v∗, u∗))

≤ θ max {d(fv, fv∗), d(fu, fu∗)}

≤ θ max {d(v, v∗), d(u, u∗) (2.4.20)

combining (2.4.19) and (2.4.20), we have

max {d(u, u∗), d(v, v∗)} ≤ θ max {d(u, u∗), d(v, v∗)}

Therefore, d (u, u∗) = 0, d (v, v∗) = 0 implies u = u∗, v = v∗.

Similarly, if (u∗, v∗) ∈ B2, then we have u = u∗ and v = v∗. Then (u, v) ∈ A2 ∩ B2 is unique coupled fixed point of covariant mappings F and f .

Finally we will show that u = v.

d (u, v) = d (F (u, v), F (v, u))

≤ θ max {d(fu, fv), d(fv, fu)}

≤ θ max {d(u, v), d(v, u)} (2.4.21)

and

d (v, u) = d (F (v, u), F (u, v))

≤ θ max {d(fv, fu), d(fu, fv)}

≤ θ max {d(v, u), d(u, v)} (2.4.22)

combining (2.4.21) and (2.4.22), we get

max {d(u, v), d(v, u)} ≤ θ max {d(u, v), d(v, u)}

< max {d(u, v), d(v, u)}

therefore, d (u, v) = 0, d (v, u) = 0 ⇒ u = v. Hence (u, u) is the common fixed point of F and f .

Corollary 2.1 Let (A, B, d) be a complete bipolar metric spaces, suppose that F:(A2, B2) ⇉ (A, B) be a covariant mapping satisfying:(2.1.1) d(F (a, b), F (p, q)) ≤ θ max {d(a, p), d(b, q)} for all a, b ∈ A and p, q ∈ B with θ ∈ (0, 1).

Then the mapping F : A2 ∪ B2 → A ∪ B has a unique coupled fixed point of the form (u, u).

Example 2.5 Let Um(R) and Lm(R) be the set of all m×m upper andlower triangular matrices over R. Define

d : Um(R) × Lm(R) → [0, ∞) as

$$d(P, Q) = \sum_{i,j=1}^{m}|p_{ij} - q_{ij}|$$

for all P = (pij)m×m ∈ Um(R) and Q = (qij)m×m ∈ Lm(R).

Then obviously (Um(R), Lm(R), d) is a Bipolar-metric space. Define F : A2 ∪ B2 → A ∪ B as F (P, Q) = $(\frac{p_{ij}}{8}+ \frac{q_{ij}}{4})$m×m and where(P = (pij)m×m, Q = (qij)m×m) ∈ Um(R)2 ∪ Lm(R)2 also f : A ∪ B → A ∪ B as f (P) = (2pij)m×m where

P = (pij)m×m ∈ Um(R) ∪ Lm(R).

Consider,

$d(F\ (P,Q), F\ (U,V\ )$

$$= d\ (\ (\tfrac{p_{ij}}{8} + \tfrac{q_{ij}}{4})m \times m, (\tfrac{u_{ij}}{8} + \tfrac{v_{ij}}{4})m \times m)$$

$$= \sum_{i,j=1}^{m}\left|\left(\frac{p_{ij}}{8} + \frac{q_{ij}}{4}\right) - \left(\frac{u_{ij}}{8} + \frac{v_{ij}}{4}\right)\right|$$

$$\leq \sum_{i,j=1}^{m}\left|\left(\frac{p_{ij}}{8} - \frac{u_{ij}}{8}\right) + \left(\frac{q_{ij}}{4} - \frac{v_{ij}}{4}\right)\right|$$

$$\leq \frac{1}{8}\sum_{i,j=1}^{m}|p_{ij} - u_{ij}| + \frac{1}{4}\sum_{i,j=1}^{m}|q_{ij} - v_{ij}|$$

$$\leq \frac{1}{16}\sum_{i,j=1}^{m}|2p_{ij} - 2u_{ij}| + \frac{1}{8}\sum_{i,j=1}^{m}|2q_{ij} - 2v_{ij}|$$

$$\leq \frac{1}{16}\max\{d(fP, fU), d(fQ, fV)\}$$

clearly F and f are satisfies all the conditions of Theorem 2.4 and (0m×m, 0m×m) is unique coupled fixed point.

Definition 2.6 Let (A, B, d) be a bipolar metric space, F : (A × B, B × A) ⇉ (A, B) and f : (A, B) ⇉ (A, B) be two covariant mappings. An element (a, p) is said to be a coupled coincident point of F and f. If F (a, p) = fa and F (p, a) = fp.

Definition 2.7 Let (A, B, d) be a bipolar metric space, F : (A × B, B × A) ⇉ (A, B) and f : (A, B) ⇉ (A, B) be two covariant mappings. An element (a, p) is said to be a common coupled fixed point of F and f. If F (a, p) = fa = a and F (p, a) = fp = p.

Definition 2.8 Let (A, B, d) be a bipolar metric space, F : (A × B, B × A) ⇉ (A, B) and f : (A, B) ⇉ (A, B) be two covariant mappings are called ω-compatible if f (F (a, p)) = F (fa, fp) and f (F (p, a)) = F (fp, fa) whenever F (a, p) = fa and F (p, a) = fp.

Theorem 2.9 Let (A, B, d) be a bipolar metric spaces, suppose that F : (A × B, B × A) ⇉ (A, B) and f : (A, B) ⇉ (A, B) be a covariant mappings satisfying:

(2.9.1) d (F (a, p), F (q, b)) ≤ θ max {d(fa, fq), d(fb, fp), }

for all a, b ∈ A and p, q ∈ B with θ ∈ (0, 1)

(2.9.2) F ((A × B) ∪ (B × A)) ⊆ f (A ∪ B).

(2.9.3) Either (F, f) is ω-compatible.

(2.9.4) f (A ∪ B) is complete.

Then the mappings F : (A × B) ∪ (B × A) → A ∪ B and f : A ∪ B → A ∪ B have a unique common fixed point of the form (u, u).

Corollary 2.2 Let (A, B, d) be a complete bipolar metric spaces, suppose that F : (A × B, B × A) ⇉(A, B) be a covariant mapping satisfying:

(2.2.1)d (F (a, p), F (q, b)) ≤ θ max {d(a, q), d(b, p)}

for all a, b ∈ A and p, q ∈ B with θ ∈ (0, 1).

Then the mapping F : (A × B) ∪ (B × A) → A ∪ B has a unique coupled fixed point of the form (u, u).

APPLICATION TO THE EXISTENCE OF SOLUTIONS OF INTEGRAL EQUATIONS

The coupled fixed point theorem proved here pave the way for application on complete bipolar metric spaces to prove the existence and uniqueness of a solution for a Fredholm nonlinear integral equation.

Theorem 3.1 Let us Consider the integral equation α(v) = $\int_{E1\cup E2}(K1\,(v,\vartheta) + K2(v,\vartheta))\,(H)d\vartheta + F(v)$, Here, H = f (ϑ, α(ϑ)) + g(ϑ, α(ϑ))) Where (v, ϑ) $\in E_1^2 \cup E_2^2$ and E1 ∪ E2 is a Lebesgue measurable set.

Suppose,

i. $K_1 : E_1^2 \cup E_2^2 \to [0,\infty), K_2 : E_1^2 \cup E_2^2 \to (-\infty, 0]$ and F ∈ L∞(E1) ∪ L∞(E2), f, g : (E1 ∪ E2) × R → R are integrable.

ii. There exists i, j ∈ (0, $\frac{1}{2}$) such that 0 ≤ f (v, α) − f (v, β) ≤ i(α − β), −j(α − β) ≤ g(v, α) − g(v, β) ≤ 0 for v ∈ E1 ∪ E2 and α, β ∈ R

iii. $\left\|\int_{E1\cup E2}(K1\,(v,\vartheta) + K2(v,\vartheta))d\vartheta\right\| \leq 1$

$i.e \sup_{v\in E1\cup E2}\int_{E1\cup E2}|K_1(v,\vartheta) - K_2(v,\vartheta)|d\vartheta \leq 1,$

for (v, ϑ) $\in E_1^2 \cup E_2^2$

then the equation has a unique solutions in L∞(E1)2 ∪ L∞(E2)2.

Proof. Let U = L∞(E1) and V = L∞(E2) be two normed linear spaces, where E1, E2 are Lebesgue measurable sets and m(E1 ∪ E2) < ∞. Let d : U × V → [0, +∞) be defined as d(Ω, Ψ) = ||Ω − Ψ||∞ for all (Ω, Ψ) ∈ U × V . Then (U, V, d) is a complete bipolar metric space. Define S : U 2 ∪ V 2 → U ∪ V by

$$S(\alpha,\beta)(v) = \int_{E1\cup E2} K1\,(v,\vartheta)\left(f(\vartheta,\alpha(v)) + g(\vartheta,\beta(v))\right)d\vartheta$$

$$+\int_{E1\cup E2} K2\,(v,\vartheta)\left(f\big(\vartheta,\beta(v)\big)+g\big(\vartheta,\alpha(v)\big)\right)d\vartheta+F(v), v\epsilon E_1 \cup E_2$$

Now we have, d (S(α, β), S(κ, ξ)) = ||S(α, β) − S(κ, ξ)||∞.

Let us first evaluate the following expression:

$$|S(\alpha,\beta)-\ S(\kappa,\xi)(v)|$$

$$= \left|\begin{matrix}\int_{E1\cup E2} K1\,(v,\vartheta)\left(f\big(\vartheta,\alpha(v)\big)+g\big(\vartheta,\beta(v)\big)\right)d\vartheta \\ \int_{E1\cup E2} K2\,(v,\vartheta)\left(f\big(\vartheta,\beta(v)\big)+g\big(\vartheta,\alpha(v)\big)\right)d\vartheta - \\ \int_{E1\cup E2} K1\,(v,\vartheta)\left(f\big(\vartheta,\kappa(v)\big)+g\big(\vartheta,\xi(v)\big)\right)d\vartheta - \\ \int_{E1\cup E2} K2\,(v,\vartheta)\left(f\big(\vartheta,\xi(v)\big)+g\big(\vartheta,\kappa(v)\big)\right)d\vartheta\end{matrix}\right|$$

$$= \left|\int_{E_1\cup E_2} K_1(v,\vartheta)(\psi)d\vartheta\right| + \left|\int_{E_1\cup E_2} K_2(v,\vartheta)(\phi)d\vartheta\right|$$

$$\le \int_{E_1\cup E_2} K_1(v,\vartheta)\,|\psi|d\vartheta + \int_{E_1\cup E_2} K_2(v,\vartheta)\,|\phi|d\vartheta$$

$$\le (i\|\alpha-\kappa\|_\infty + j\|\beta-\xi\|_\infty)\int_{E1\cup E2} (K1\,(v,\vartheta)-K2(v,\vartheta))d\vartheta$$

Where

ψ = f (ϑ, α(v)) − f (ϑ, κ(v)) + g (ϑ, β(v)) − g(ϑ, ξ(v)),

ϕ = f (ϑ, β(v)) − f (ϑ, ξ(v)) + g(ϑ, α(v)) − g(ϑ, κ(v))

Then,

d (S(α, β), S(κ, ξ)) = ||S(α, β) − S(κ, ξ)||∞

≤ (i||α − κ||∞ + j||β − ξ||∞) $\left\|\int_{E1\cup E2}(K1\,(v,\vartheta)-K2(v,\vartheta))d\vartheta\right\|$

≤ (i||α − κ||∞ + j||β−ξ||∞) $\sup_{v\in E1\cup E2}\int_{E1\cup E2}|K_1(v,\vartheta)-\ K_2(v,\vartheta)|d\vartheta$

≤ i||α − κ||∞ + j||β − ξ||∞

≤ θ max {||α − κ)||∞, ||β − ξ||∞}

≤ θ max {d (α, κ), d(β, ξ)}

Hence, applying our Corollary 2.1 we get the desired result.

APPLICATION TO HOMOTOPY

Theorem 4.1 Let (A, B, d) be complete bipolar metric space, (U, V) be an open subset of (A, B) and $(\overline{U},\overline{V})$ be closed subset of (A, B) such that (U, V) ⊆ $(\overline{U},\overline{V})$. Suppose H : $(\overline{U}^2,\overline{V}^2)$×[0, 1] →A ∪ B be an operator with following conditions are satisfying,

(4.1.1) u ≠ H(u, v, κ) and v ≠ H(v, u, κ)for each u, v ∈ ∂U ∪ ∂V and κ ∈ [0, 1] (Here ∂U ∪ ∂V is boundary of U ∪ V in A ∪ B)

(4.1.2) d (H(u, v, κ), H(x, y, κ)) ≤ θ max {d(u, x), d(v, y)}

for all u, v ∈$\overline{U}$, x, y ∈$\overline{V}$ and κ ∈ [0, 1], θ ∈ (0, 1)

(4.1.3) ∃ M ≥ 0 ∋ d(H(u, v, κ), H(x, y, ζ)) ≤ M |κ − ζ|

for every u, v ∈$\overline{U}$ and x, y ∈$\overline{V}$ and κ, ζ ∈ [0, 1].

Then H(., 0) has a fixed point ⇔ H(., 1) has a fixed point.

Proof. Let the set

X = {κ ∈ [0, 1] : u = H(u, v, κ), v = H(v, u, κ) for some (u, v) ∈ U 2 ∪ V 2}.

Y = {ζ ∈ [0, 1] : x = H(x, y, ζ), y = H(y, x, ζ) for some (x, y) ∈ U 2 ∪ V 2}.

Since H(., 0) has a fixed point in U 2 ∪ V 2, so (0, 0) ∈ X2 ∩ Y 2.

Now we show that X2 ∩ Y 2 is both closed and open in [0, 1] and hence by the connectednessX = Y = [0, 1].

Let ({κn}∞n=1 , {ζn}∞n=1) ⊆ (X, Y) with (κn, ζn) → (κ, ζ) ∈ [0, 1] as n → ∞.We must show that (κ, ζ) ∈ X2 ∩ Y 2.

Since (κn, ζn) ∈ (X, Y) for n = 0, 1, 2, 3, · · · , there exists bisequences (un, xn), (vn, yn) withun+1 = H(un, vn, κn), vn+1 = H(vn, un, κn) and xn+1 = H(xn, yn, ζn), yn+1 = H(yn, xn, ζn).

Consider,

d(un, xn+1) = d(H(un−1, vn−1, κn−1), H(xn, yn, ζn))

≤ θ max {d(un−1, xn), d(vn−1, yn)} (4.1.4)

And

d(vn, yn+1) = d(H(vn−1, un−1, κn−1), H(yn, xn, ζn))

≤ θ max {d(vn−1, yn), d(un−1, xn)} (4.1.5)

Combining (4.1.4) and (4.1.5)

max {d(un, xn+1), d(vn, yn+1)}

≤ θ max {d(un−1, xn), d(vn−1, yn)}

≤ θ2 max {d(un−2, xn−1), d(vn−2, yn−1)}

≤ θn max {d(u0, x1), d(v0, y1)} .

Thus

d(un, xn+1) ≤ θn max {d(u0, x1), d(v0, y1)} ,

d(vn, yn+1) ≤ θn max {d(u0, x1), d(v0, y1)} (4.1.6)

Similarly, we can prove

d(un+1, xn) ≤ θn max {d(u1, x0), d(v1, y0)} ,

d(vn+1, yn) ≤ θn max {d(u1, x0), d(v1, y0)} (4.1.7)

Also

d(un, xn) ≤ θn max {d(u0, x0), d(v0, y0)} ,

d(vn, yn) ≤ θn max {d(u0, x0), d(v0, y0)} (4.1.8)

for each n, m ∈ N, n < m Using the property (B4) and (4.1.6), (4.1.7), (4.1.8), we have

d(un, xm) + d(vn, ym)

≤ (d(un, xn+1) + d(vn, yn+1)) + (d(un+1, xn+1)

+ d(vn+1, yn+1)) + · · · · · ·+(d(um−1, xm−1)

+ d(vm−1, ym−1)) + (d(um−1, xm) + d(vm−1, ym))

≤ 2 θn max {d(u0, x1), d(v0, y1)} + M |κn+1 − ζn+1| + · · ·

+M |κm−1 − ζm−1| + 2θm max {d(u0, x1),

d(v0, y1)} → 0 as n, m → ∞.

It follows $\lim_{n\to\infty}(d(un,xm)\ +\ d(vn,ym))\ =\ 0$

Similarly, we can show $\lim_{n\to\infty}(d(um,xm)\ +\ d(vm,yn))\ =\ 0$

Therefore, (un, xn) and (vn, yn) are Cauchy bisequence in (U, V). By completeness, there exist ξ, ν ∈ U and δ, η ∈ V with

$$\lim_{n\to\infty}u_n=\delta, \lim_{n\to\infty}v_n=\eta, \lim_{n\to\infty}x_n=\xi and \lim_{n\to\infty}y_n=v \qquad (4.1.9)$$

Now consider

d(H(ξ, ν, κ), δ)

≤ d(H(ξ, ν, κ), xn+1) + d(un+1, xn+1) + d(un+1, δ)

≤ d(H(ξ, ν, κ), H(xn, yn, ζn))

+d(H(un, vn, κn), H(xn, yn, ζn)) + d(un+1, δ)

≤ θ max {d(un, xn), d(vn, yn)} + M |κn − ζn| + d(un+1, δ)

< max {d(un, xn), d(vn, yn)}+M |κn − ζn|

+ d(un+1, δ) → 0 as n → ∞.

It follows d(H(ξ, ν, κ), δ) = 0 implies H(ξ, ν, κ) = δ. Similarly we get H(ν, ξ, κ) = η and

H(δ, η, ζ) = ξ, H(η, δ, ζ) = ν. On the other hand from (4.1.9), we get

$$d(\xi,\delta) = d\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} d(u_n, x_n) = 0$$

$$d(\nu,\eta) = d\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} d(v_n, y_n) = 0$$

Therefore, ξ = δ and ν = η and hence κ = ζ.

Thus (κ, ζ) ∈ X2 ∩ Y 2. Clearly X2 ∩ Y 2 is closed in [0, 1].

Let (κ0, ζ0) ∈ (X, Y), then there exist bisequences (u0, x0), (v0, y0) with

u0 = H(u0, v0, κ0), v0 = H(v0, u0, κ0) and x0 = H(x0, y0, ζ0), y0 = H(y0, x0, ζ0).

Since U 2 ∪ V 2 is open, then there exist r > 0 such that

Xd(u0, r) ⊆ U 2 ∪ V 2 and Xd(v0, r) ⊆ U 2 ∪ V 2 and Xd(x0, r) ⊆ U 2 ∪ V 2 and Xd(y0, r) ⊆ U 2 ∪ V 2.

Choose $\kappa \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon), \zeta \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}, |\zeta - \kappa_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$

and $|\kappa_0 - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$.

Then for

x ∈ BX∪Y (u0, r) = {x, x0 ∈ V/d(u0, x) ≤ r + d(u0, x0)},

y ∈ BX∪Y (v0, r) = {y, y0 ∈ V/d(v0, y) ≤ r + d(v0, y0)} and

u ∈ BX∪Y (r, x0) = {u, u0 ∈ U/d(u, x0) ≤ r + d(u0, x0)},

v ∈ BX∪Y (r, y0) = {v, v0 ∈ U/d(v, y0) ≤ r + d(v0, y0)} .

Also

d(H(u, v, κ), x0) = d(H(u, v, κ), H(x0, y0, ζ0))

≤ d(H(u, v, κ), H(x, y, ζ0)) + d(H(u0, v0, κ), H(x, y, ζ0))

+ d(H(u0, v0, κ),H(x0, y0, ζ0))

Letting n → ∞, we get

d(H(u, v, κ), x0) ≤ θ max {d(u0, x), d(v0, y)}(4.1.10)

Similarly, we can prove

d(H(v, u, κ), y0) ≤ θ max {d(v0, y), d(u0, x)}(4.1.11)

Combining 4.1.10) and 4.1.11), we get

max {d(H(u, v, κ), x0), d(H(u, v, κ), x0)}

≤ θ max {d(u0, x), d(v0, y)}

< max {d(u0, x), d(v0, y)}

≤ max {d(u0, x0) + r, d(v0, y0) + r}

Thus

d(H(u, v, κ), x0)

≤ d(u0, x0) + r, d(H(u, v, κ), x0)

≤ d(v0, y0) + r.

Similarly, d(u0, H(x, y, ζ),) ≤ d(u, x0) ≤ r + d(u0, x0) and

d(v0, H(y, x, ζ),) ≤ d(v, y0) ≤ r + d(v0, y0).

On the other hand

d(u0, x0) = d(H(u0, v0, κ0), H(x0, y0, ζ0)) ≤ M |κ0 − ζ0|

$\leq M\frac{1}{M^n} \leq \frac{1}{M^{n-1}} \to 0$ as $n \to \infty$

And

d(v0, y0) = d(H(v0, u0, κ0), H(y0, x0, ζ0))≤ M |κ0 − ζ0|

$\leq M\frac{1}{M^n} \leq \frac{1}{M^{n-1}} \to 0$ as $n \to \infty$

So u0 = x0 and v0 = y0 and hence κ = ζ.

Thus for each fixed κ ∈ (κ0 − ϵ, κ0 + ϵ),

H(., κ) : $\overline{BX \cup Y\ (u0,r)} \to \overline{BX \cup Y(u0,r)}$ and

H(., κ) : $\overline{BX \cup Y\ (v0,r)} \to \overline{BX \cup Y\ (v0,r)}$

Then all conditions of Theorem (4.1) are satisfied.

Thus we conclude that H(., κ) has a coupled fixed point in $\bar{U}$2 ∩ $\bar{V}$ 2. But this must be in U 2 ∩ V 2.

Therefore, (κ, κ) ∈ X2 ∩ Y 2 for κ ∈ (κ0 − ϵ, κ0 + ϵ). Hence (κ0 − ϵ, κ0 + ϵ) ⊆ X2 ∩ Y 2. Clearly X2 ∩ Y 2 is open in [0, 1].

To prove the reverse, we can use the similar process.

Theorem 4.2 Let (A,B, d) be complete bipolar metric space, (U, V) be an open subset of (A,B)and ($\bar{U}, \bar{V}$) be closed subset of (A,B) such that (U, V) ⊆($\bar{U}, \bar{V}$). Suppose H : ($\bar{U} X \bar{V}$) ∪ ($\bar{V}, \bar{U}$)×[0, 1] →A ∪ B be an operator with following conditions are satisfying,

(4.1.1) u ≠ H(u, v, κ) and v ≠ H(v, u, κ)for each u, v ∈ ∂U ∪ ∂V and κ ∈ [0, 1] (Here ∂U ∪ ∂Vis boundary of U ∪ V in A ∪ B)

(4.1.2) d (H(u, v, κ),H(x, y, κ)) ≤ θ max {d(u, x), d(v, y)}

for all u, v ∈$\bar{U}$, x, y ∈$\bar{V}$ and κ ∈ [0, 1], θ ∈ (0, 1)

(4.1.3) ∃ M ≥ 0 ∋ d(H(u, v, κ),H(x, y, ζ)) ≤M|κ− ζ|

for every u, v ∈$\bar{U}$ and x, y ∈$\bar{V}$ and κ, ζ ∈ [0, 1].

Then H(., 0) has a fixed point ⇐⇒ H(., 1) has a fixed point.

CONCLUSIONS

In this paper, we obtain the existence and uniqueness solution for two covariant mappings in a complete bipolar metric spaces with an example. Also, we have provided some applications to nonlinear integral equations as well as Homotopy theory by using fixed point theorems in bipolar metric spaces.

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